THE

MESSENGER OF MATHEMATICS,

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MESSENGER OF MATHEMATICS,

ON CERTAIN SUMS OF PRODUCTS OF QUANTITIES DEPENDING UPON THE DIVISORS OF A NUMBER.

By *J. W. L. Glaisher.*

Object of the paper, § 1.

§ 1. The present paper relates to the values of a system of expressions of the forms

$$f(1)f(n-1)+f(2)f(n-2)+f(3)f(n-3)\ldots+f(n-1)f(1)$$

and

$$f(1)F(n-1)+f(2)F(n-2)+f(3)F(n-3)\ldots+f(n-1)F(1),$$

where $f(n)$ and $F(n)$ are any two of the following seven quantities which depend on the divisors of n:

$\sigma(n) =$ sum of the divisors of n;

$\Delta(n) =$ " " uneven divisors;

$D(n) =$ " " even " ;

$\Delta'(n) =$ " " divisors whose conjugates are uneven;

$D'(n) =$ " " " " " " even;

$\zeta(n) = \begin{cases} \text{" " uneven divisors} \\ - \text{" " even " ;} \end{cases}$

$\zeta'(n) = \begin{cases} \text{" " divisors whose conjugates are uneven} \\ - \text{" " " " " " even.} \end{cases}$

Thus $\sigma(n) = \Delta(n) + D(n) = \Delta'(n) + D'(n),$

$$\zeta(n) = \Delta(n) - D(n),$$

and $\zeta'(n) = \Delta'(n) - D'(n).$

It is convenient, however, to first consider, in a more general form, the values of these seven quantities and the relations between them. The consideration of these more general functions will form the subject of the next five sections.

The functions $\sigma_s(n)$, $\Delta_s(n)$, $D_s(n)$, &c., §§ 2–6.

§ 2. The suffix s attached to a function will be used to denote that, instead of the simple divisors of n, the s^{th} powers of those divisors are taken in the formation of the function. Thus $f_s(n)$ is formed from the s^{th} powers of the divisors of n in exactly the same manner as $f(n)$ is formed from the simple divisors; so that, for example, $\Delta'_s(n)$ denotes the sum of the s^{th} powers of those divisors of n whose conjugates are uneven.

The absence of a suffix is therefore equivalent to a suffix unity, viz., $f(n) = f_1(n)$.

It follows immediately from the definitions that

$$\sigma_s(n) = \Delta_s(n) + D_s(n) = \Delta'_s(n) + D'_s(n),$$

$$\zeta_s(n) = \Delta_s(n) - D_s(n),$$

$$\zeta'_s(n) = \Delta'_s(n) - D'_s(n),$$

and it can be shown that, if $n = 2^p m$, where m denotes an uneven number, then

$$\sigma_s(n) = \frac{2^{ps+s} - 1}{2^s - 1}\Delta_s(n),$$

$$\Delta_s(n) = \Delta_s(n),$$

$$D_s(n) = \frac{2^{ps+s} - 2^s}{2^s - 1}\Delta_s(n),$$

$$\Delta'_s(n) = 2^{ps}\Delta_s(n),$$

$$D'_s(n) = \frac{2^{ps} - 1}{2^s - 1}\Delta_s(n),$$

$$\zeta_s(n) = -\frac{2^{ps+s} - 2^{s+1} + 1}{2^s - 1}\Delta_s(n),$$

$$\zeta'_s(n) = \frac{2^{ps+s} - 2^{ps+1} + 1}{2^s - 1}\Delta_s(n).$$

It follows from these formulæ, or can be readily shown independently, that

$$D_s(n) = 2^s D_s'(n);$$

$$D_s(n) - D_s'(n) = \Delta_s'(n) - \Delta_s(n);$$

$$D_s'(2n) = \sigma_s(n),$$

and, by considering separately the cases of n even and n uneven, we may deduce from the formulæ for $\zeta_s(n)$, &c., the relation

$$\zeta_s(n) - \zeta_s'(n) = (-)^n 2\,\{\Delta_s(n) - \Delta_s'(n)\}.$$

The above formulæ are still true when $p = 0$, in which case $D_s(m)$ and $D_s'(m)$ are zero, and

$$\sigma_s(m) = \Delta_s(m) = \Delta_s'(m) = \zeta_s(m) = \zeta_s'(m),$$

m denoting any uneven number.

§ 3. If $n = 2^p a^\alpha b^\beta c^\gamma \ldots$, where a, b, c, ... are any prime numbers different from one another, then

$$\Delta_s(n) = \frac{(a^{\alpha s+s} - 1)\,(b^{\beta s+s} - 1)\,(c^{\gamma s+s} - 1)\ldots}{(a^s - 1)\,(b^s - 1)\,(c^s - 1)\ldots}$$

$$= \Delta_s(a^\alpha)\,\Delta_s(b^\beta)\,\Delta_s(c^\gamma)\ldots .$$

It follows, therefore, from the formulæ in the last section, that if f_s denote any one of the seven functions σ_s, Δ_s, D_s, Δ'_s, D'_s, ζ_s, ζ'_s, then

$$f_s(n) = f_s(2^p)\,\Delta_s(a^\alpha)\,\Delta_s(b^\beta)\,\Delta_s(c^\gamma) \ldots\ldots\ldots \quad (1),$$

and, if $n = 2^p m_1 m_2 m_3, \ldots$ where m_1, m_2, m_3, ... are any uneven numbers which are all prime to each other,

$$f_s(n) = f_s(2^p)\,\Delta_s(m_1)\,\Delta_s(m_2)\,\Delta_s(m_3)\ldots .$$

Also, if $n = n_1 n_2 n_3, \ldots$ where n_1, n_2, n_3, ... are any numbers which are all prime to each other, $f_s(n) = f_s(n_1) f_s(n_2) f_s(n_3) \ldots$.

The most simple and rapid method of tabulating any one of the functions $f_s(n)$ is to calculate the values of

$$f_s(2^r),\ \Delta_s(3^r),\ \Delta_s(5^r),\ \Delta_s(7^r),\ \Delta_s(11^r)\ldots,$$

for $r = 1, 2, 3, \ldots$ and then to deduce the values of $f_s(n)$ corresponding to the arguments which are not powers of primes by simple multiplication by means of (1).

§ 4. The seven functions may be expressed in terms of any two of them by formulæ which do not involve p. The two quantities which the forms of the expressions in § 2 seem to suggest most naturally for this purpose are $\Delta_s(n)$ and $\Delta_s'(n)$; and, since $\Delta'_s(n) = 2^{ps}\Delta_s(n)$, we deduce at once from § 2 the formulæ

$$\sigma_s(n) = \frac{2^s\Delta'_s(n) - \Delta_s(n)}{2^s - 1},$$

$$\Delta_s(n) = \Delta_s(n),$$

$$D_s(n) = \frac{2^s\Delta'_s(n) - 2^s\Delta_s(n)}{2^s - 1}$$

$$\Delta'_s(n) = \Delta'_s(n),$$

$$D'_s(n) = \frac{\Delta'_s(n) - \Delta_s(n)}{2^s - 1},$$

$$\zeta_s(n) = \frac{(2^{s+1} - 1)\,\Delta_s(n) - 2^s\Delta'_s(n)}{2^s - 1},$$

$$\zeta_s'(n) = \frac{(2^s - 2)\,\Delta'_s(n) + \Delta_s(n)}{2^s - 1}.$$

§ 5. The formulæ in the last section involve the denominator $2^s - 1$; and on this account they are not so simple as the corresponding equations when $\sigma_s(n)$ and $D'_s(n)$, or $\sigma_s(n)$ and $\Delta'_s(n)$, or $D'_s(n)$ and $\Delta'_s(n)$ are selected as the two quantities in terms of which the others are expressed.

These systems of formulæ, which are free from the presence of denominators, are as follows:

$$\begin{aligned}
\sigma_s(n) &= \sigma_s(n) \\
\Delta_s(n) &= \sigma_s(n) - 2^sD'_s(n), \\
D_s(n) &= 2^sD'_s(n), \\
\Delta'_s(n) &= \sigma_s(n) - D'_s(n), \\
D'_s(n) &= D'_s(n), \\
\zeta_s(n) &= \sigma_s(n) - 2^{s+1}D'_s(n), \\
\zeta_s'(n) &= \sigma_s(n) - 2\,D'_s(n);
\end{aligned}$$

$$
\begin{aligned}
\sigma_s(n) &= \sigma_s(n),\\
\Delta_s(n) &= -\ (2^s-1)\,\sigma_s(n) + 2^s\,\Delta'_s(n),\\
D_s(n) &= 2^s\sigma_s(n) - 2^s\,\Delta'_s(n),\\
\Delta'_s(n) &= \Delta'_s(n),\\
D'_s(n) &= \sigma_s(n) - \Delta'_s(n),\\
\zeta_s(n) &= -\ (2^{s+1}-1)\,\sigma_s(n) + 2^{s+1}\Delta'_s(n),\\
\zeta_s'(n) &= -\,\sigma_s(n) + 2\Delta'_s(n);
\end{aligned}
$$

$$
\begin{aligned}
\sigma_s(n) &= \Delta'_s(n) + D'_s(n),\\
\Delta_s(n) &= \Delta'_s(n) - (2^s-1)\,D'_s(n),\\
D_s(n) &= 2^sD'_s(n),\\
\Delta'_s(n) &= \Delta'_s(n),\\
D'_s(n) &= D'_s(n),\\
\zeta_s(n) &= \Delta'_s(n) - (2^{s+1}-1)\,D'_s(n),\\
\zeta_s'(n) &= \Delta'_s(n) - D'_s(n).
\end{aligned}
$$

§ 6. The simplest of these three systems is the first, in which $\sigma_s(n)$ and $D'_s(n)$ are taken as the two functions in terms of which the others are expressed; and, with reference to this system, it is to be noticed that, since

$$D'_s(2n) = \sigma_s(n),$$

we have

$$D'_s(n) = \sigma_s(\tfrac{1}{2}n),$$

if we define $\sigma_s(p)$ to denote zero when p is not integral.

Defining the function $\sigma_s(n)$ in this manner *i.e.* so that $\sigma_s(n)$ denotes the sum of the s^{th} powers of the divisors of n, if n be an integer, and denotes zero if n be fractional, we may therefore express all the functions in terms of the σ_s-function by means of the formulæ

$$
\begin{aligned}
\sigma^s(n) &= \sigma_s(n),\\
\Delta_s(n) &= \sigma_s(n) - 2^s\,\sigma_s(\tfrac{1}{2}n),\\
D_s(n) &= 2^s\,\sigma_s(\tfrac{1}{2}n),\\
\Delta'_s(n) &= \sigma_s(n) - \sigma_s(\tfrac{1}{2}n),\\
D'_s(n) &= \sigma_s(\tfrac{1}{2}n),\\
\zeta_s(n) &= \sigma^s(n) - 2^{s+1}\sigma_s(\tfrac{1}{2}n),\\
\zeta_s'(n) &= \sigma^s(n) - 2\sigma^s(\tfrac{1}{2}n).
\end{aligned}
$$

The results referred to in § 1 involve only the functions $\sigma(n)$, $\Delta(n)$, &c. and $\sigma_3(n)$, $\Delta_3(n)$, &c.. viz. the general functions $\sigma_s(n)$, $\Delta_s(n)$, &c. do not occur, but only the special cases $s=1$ and $s=3$.

The case $s=1$, § 7.

§ 7. When $s=1$, the functions $\zeta_s'(n)$ and $\Delta_s'(n)$ become equal, and we have

$$\zeta'(n)=\Delta(n).$$

By putting $s=1$ in the general formulæ of the last two sections, we obtain the following systems of equations which are useful for purposes of reference.

$$\begin{aligned}
\sigma(n) &= \sigma(n) &&= \sigma(n)\\
\Delta(n) &= \sigma(n)-2D'(n) &&= \sigma(n)+2\Delta'(n),\\
D(n) &= 2D'(n) &&= 2\sigma(n)-2\Delta'(n),\\
\Delta'(n) &= \sigma(n)-D'(n) &&= \Delta'(n),\\
D'(n) &= D'(n) &&= \sigma(n)-\Delta'(n),\\
\zeta(n) &= \sigma(n)-4D'(n) &&= -3\sigma(n)+4\Delta'(n),\\
\zeta'(n) &= \sigma(n)-2D'(n) &&= -\sigma(n)+2\Delta'(n);
\end{aligned}$$

$$\begin{aligned}
\sigma(n) &= \Delta'(n)+D'(n) &&= 2\Delta'(n)-\Delta(n),\\
\Delta(n) &= \Delta'(n)-D'(n) &&= \Delta(n),\\
D(n) &= 2D'(n) &&= 2\Delta'(n)-2\Delta(n),\\
\Delta'(n) &= \Delta'(n) &&= \Delta'(n)\\
D'(n) &= D'(n) &&= \Delta'(n)-\Delta(n),\\
\zeta(n) &= \Delta'(n)-3D'(n) &&= -2\Delta'(n)+3\Delta(n),\\
\zeta'(n) &= \Delta'(n)-D'(n) &&= \Delta(n).
\end{aligned}$$

These equations express all the functions in terms of $\sigma(n)$ and $D'(n)$, of $\sigma(n)$ and $\Delta'(n)$, of $\Delta'(n)$ and $D'(n)$, and of $\Delta'(n)$ and $\Delta(n)$.

Replacing $D'(n)$ by $\sigma(\tfrac{1}{2}n)$ in the first system of equations we find

$$\begin{aligned}
\sigma(n) &= \sigma(n), & \zeta(n) &= \sigma(n)-4\sigma(\tfrac{1}{2}n),\\
D(n) &= 2\sigma(\tfrac{1}{2}n), & \Delta(n) &= \sigma(n)-2\sigma(\tfrac{1}{2}n)=\zeta'(n),\\
D'(n) &= \sigma(\tfrac{1}{2}n), & \Delta'(n) &= \sigma(n)-\sigma(\tfrac{1}{2}n).
\end{aligned}$$

The case $s=3$, § 8.

§ 8. For the sake of reference, the following equations which express $\sigma_3(n)$, $\Delta_3(n)$, &c. in terms of $\sigma_3(n)$ and $D'_3(n)$ are also given.

$$
\begin{aligned}
\sigma_3(n) &= \sigma_3(n), \\
\Delta_3(n) &= \sigma_3(n) - 8D'_3(n), \\
D_3(n) &= 8D'_3(n), \\
\Delta'_3(n) &= \sigma_3(n) - D'_3(n), \\
D'_3(n) &= D'_3(n), \\
\zeta_3(n) &= \sigma_3(n) - 16D'_3(n), \\
\zeta_3'(n) &= \sigma_3(n) - 2D'_3(n).
\end{aligned}
$$

Since $D'_3(n) = \sigma_3(\tfrac{1}{2}n)$, we have also

$$
\begin{aligned}
\sigma_3(n) &= \sigma_3(n), & \zeta_3(n) &= \sigma_3(n) - 16\sigma_3(\tfrac{1}{2}n), \\
D_3(n) &= 8\sigma_3(\tfrac{1}{2}n), & \Delta_3(n) &= \sigma_3(n) - 8\sigma_3(\tfrac{1}{2}n), \\
& & \zeta_3'(n) &= \sigma_3(n) - 2\sigma_3(\tfrac{1}{2}n), \\
D'_3(n) &= \sigma_3(\tfrac{1}{2}n), & \Delta'_3(n) &= \sigma_3(n) - \sigma_3(\tfrac{1}{2}n).
\end{aligned}
$$

Sums of products, §§ 9–15.

§ 9. Consider now the expression

$$\Sigma_{r=1}^{r=n} f(r)\,F(n-r) = \Sigma_{r=1}^{r=n} F(r)\,f(n-r)$$

$$=f(1)\,F(n-1)+f(2)\,F(n-2)+f(3)\,F(n-3)\ldots+f(n-1)\,F(1),$$

where f and F denote any two of the functions

$$\sigma,\ \Delta,\ D,\ \Delta',\ D',\ \zeta,$$

the function ζ' being omitted because $\zeta'(n) = \Delta(n)$.

Suppose f and F expressed in terms of σ and D'; then we have

$$
\begin{aligned}
f(n) &= \alpha\,\sigma(n) + \beta\,D'(n), \\
F(n) &= \alpha'\sigma(n) + \beta' D'(n),
\end{aligned}
$$

where $\alpha = 0$ or 1 and $\beta = 0, \pm 1, \pm 2$ or -4.

Therefore

$$
\begin{aligned}
f(r)\,F(n-r) = {} & \alpha\alpha'\sigma(r)\,\sigma(n-r) + \alpha\beta'\sigma(r)\,D'(n-r) \\
& + \alpha'\beta\sigma(n-r)\,D'(r) + \beta\beta' D'(r)\,D'(n-r),
\end{aligned}
$$

whence, by putting $r=1, 2, \ldots, n-1$, and adding the terms,

$$\Sigma_{r=1}^{r=n} f(r)\, F(n-r) = \alpha\alpha' \Sigma_{r=1}^{r=n} \sigma(n)\, \sigma(n-r)$$
$$+ (\alpha\beta' + \alpha'\beta)\, \Sigma_{r=1}^{r=n} \sigma(r)\, D'(n-r) + \beta\beta' \Sigma_{r=1}^{r=n} D'(r)\, D'(n-r).$$

Denoting, for brevity,

$$f(1)\,F(n-1) + f(2)\,F(n-2) + f(3)\,F(n-3) \ldots + f(n-1)\,F(1),$$

by ΣfF, so that $\Sigma fF = \Sigma Ff$, the equation just found may be written

$$\Sigma fF = \alpha\alpha' \Sigma\sigma\sigma + (\alpha\beta' + \alpha'\beta)\, \Sigma\sigma D' + \beta\beta' \Sigma D'D'.$$

If now we put

$$\begin{aligned} \Sigma\sigma\sigma &= (ss), \\ \Sigma\sigma D' &= (sd'), \\ \Sigma D'D' &= (d'd'), \end{aligned}$$

then $$\Sigma fF = \alpha\alpha'\,(ss) + (\alpha\beta' + \alpha'\beta)\,(sd') + \beta\beta'\,(d'd').$$

The expression which forms the right-hand member of this equation is the same as that which we should obtain by multiplying together the two factors

$$\alpha s + \beta d' \quad \text{and} \quad \alpha' s + \beta' d'$$

and replacing ss, sd', $d'd'$ by the quantities (ss), (sd'), $(d'd')$; it is thus evident that, in order to express ΣfF in terms of $\Sigma\sigma\sigma$, $\Sigma\sigma D'$, and $\Sigma D'D'$, it is only necessary to multiply together the expressions for f and F in terms of σ and D'.

For example, since

$$\Delta(n) = \sigma(n) - 2D'(n),$$

and

$$\zeta(n) = \sigma(n) - 4D'(n),$$

we find

$$\begin{aligned} \Sigma\Delta\zeta &= (s - 2d')(s - 4d') \\ &= (ss) - 6\,(sd') + 8\,(d'd'). \end{aligned}$$

§ 10. Expressing in this manner the twenty-one sums of products that can be found from the six quantities σ, Δ, D, Δ', D', ζ, we obtain the equations:

$$\begin{aligned} \Sigma\sigma\sigma &= (ss), \\ \Sigma\Delta\Delta &= (ss) - 4\,(sd') + 4\,(d'd'), \\ \Sigma DD &= 4\,(d'd'), \\ \Sigma\Delta'\Delta' &= (ss) - 2\,(sd') + (d'd'), \\ \Sigma D'D' &= (d'd'), \end{aligned}$$

$$\begin{aligned}
\Sigma\zeta\zeta &= (ss) - 8\,(sd') + 16\,(d'd'),\\
\Sigma\Delta\Delta' &= (ss) - 3\,(sd') + 2\,(d'd'),\\
\Sigma DD' &= 2\,(d'd'),\\
\Sigma\sigma\Delta &= (ss) - 2\,(sd'),\\
\Sigma\sigma D &= 2\,(sd'),\\
\Sigma\sigma\Delta' &= (ss) - (sd'),\\
\Sigma\sigma D' &= (sd'),\\
\Sigma\sigma\zeta &= (ss) - 4\,(sd'),\\
\Sigma\Delta D &= 2\,(sd') - 4\,(d'd'),\\
\Sigma\Delta D' &= (sd') - 2\,(d'd'),\\
\Sigma\Delta\zeta &= (ss) - 6\,(sd') + 8\,(d'd'),\\
\Sigma D\Delta' &= 2\,(sd') - 2\,(d'd'),\\
\Sigma D\zeta &= 2\,(sd') - 8\,(d'd'),\\
\Sigma\Delta' D' &= (sd') - (d'd'),\\
\Sigma\Delta'\zeta &= (ss) - 5\,(sd') + 4\,(d'd'),\\
\Sigma D'\zeta &= (sd') - 4\,(d'd').
\end{aligned}$$

It is scarcely necessary to remark that these 21 product-sums might be expressed in terms of any two of the functions by a process exactly similar to that which has just been used to express them in terms of σ and D'.

§ 11. The equations obtained in the last section give the values of the 21 product-sums when the values of (ss), (sd'), $(d'd')$ are known.

The value of (ss) was found on p. 160 of vol. XIV. of the *Messenger*, where it was shown that

$$12\Sigma\sigma\sigma = 5\sigma_3(n) - 6n\sigma(n) + \sigma(n).$$

The value of $(d'd')$ may be at once deduced from this formula, for

$$\begin{aligned}
\Sigma D'D' &= D'(1)\,D'(n-1) + D'(2)\,D'(n-2)\ldots + D'(n-1)\,D'(1)\\
&= D'(2)\,D'(n-2) + D'(4)\,D'(n-4)\ldots + D'(n-2)\,D'(2),
\end{aligned}$$

since $D(r)$ is zero when r is uneven.

Thus $\Sigma D'D'$ is zero unless n is even, and, in this case, since $D'(2m) = \sigma(m)$,

$$\begin{aligned}\Sigma D'D' &= \sigma(1)\,\sigma(\tfrac{1}{2}n-1) + \sigma(2)\,\sigma(\tfrac{1}{2}n-2) \ldots + \sigma(\tfrac{1}{2}n-1)\,\sigma(1)\\ &= \tfrac{1}{12}\{5\sigma_3(\tfrac{1}{2}n) - 3n\sigma(\tfrac{1}{2}n) + \sigma(\tfrac{1}{2}n)\}\\ &= \tfrac{1}{12}\{5D'_3(n) - 3nD'(n) + D'(n)\}.\end{aligned}$$

This expression vanishes when n is uneven and gives therefore the correct value of $\Sigma D'D'$ for all values of n.

The value of (sd') may be deduced from the formula for $\Sigma\Delta'\Delta'$ given on p. 104 of vol. XIV., viz.

$$4\Sigma\Delta'\Delta' = \Delta'_3(n) - n\Delta'(n);$$

for, by the last section,

$$\Sigma\Delta'\Delta' = (ss) - 2(sd') + (d'd'),$$

whence

$$\begin{aligned}24(sd') = {} & 12(ss) + 12(d'd') - 12\Sigma\Delta'\Delta'\\ = {} & 5\sigma_3(n) - 6n\sigma(n) + \sigma(n)\\ & + 5D'_3(n) - 3nD'(n) + D'(n)\\ & - 3\sigma_3(n) + 3D'_3(n) + 3n\sigma(n) - 3nD'(n)\\ = {} & 2\sigma_3(n) + 8D'_3(n) - 3n\sigma(n) - 6nD'(n) + \sigma(n) + D'(n).\end{aligned}$$

Thus we have finally

$$\begin{aligned}24(ss) &= 10\sigma_3(n) - 12n\sigma(n) + 2\sigma(n),\\ 24(sd') &= 2\sigma_3(n) - 3n\sigma(n) + \sigma(n)\\ &\quad + 8D'_3(n) - 6nD'(n) + D'(n),\\ 24(d'd') &= 10D'_3(n) - 6nD'(n) + 2D'(n).\end{aligned}$$

§ 12. On p. 104 the value of $\Sigma\zeta\zeta$ is also given, the formula being

$$4\Sigma\zeta\zeta = -\zeta_3(n) + (2n-1)\,\zeta(n).$$

This equation may be used as a means of verifying the values of (ss), (sd'), $(d'd')$ for, by § 10,

$$\Sigma\zeta\zeta = (ss) - 8(sd') + 16(d'd'),$$

and, multiplying by 24, and substituting for $24(ss)$, $24(sd')$, $24(d'd')$ their values, we find

$$24\Sigma\zeta\zeta = -6\sigma_3(n) + 96D'_3(n) + 12n\sigma(n) - 6\sigma(n) - 48nD'(n) + 24D'(n),$$

which agrees with the above value of $\Sigma\zeta\zeta$, since, by §§ 7 and 8,

$$\zeta_3(n) = \sigma_3(n) - 16D'_3(n),$$
$$\zeta\ (n) = \sigma\ (n) - \ \ 4D'\ (n).$$

§ 13. The values of the 21 products-sums are given below. They were deduced from the formulæ in § 10 by substituting for (ss), (sd'), $(d'd')$ their values in terms of $\sigma_3(n)$, $D'_3(n)$, $\sigma(n)$, $D(n)$ given in § 11. It is to be remembered that

$$\Sigma fF = \Sigma Ff = \Sigma_{r=1}^{r=n} f(r)\, F(n-r) = \Sigma_{r=1}^{r=n} F(r)\, f(n-r)$$
$$= f(1)F(n-1) + f(2)F(n-2) + f(3)F(n-3) \ldots + f(n-1)F(1),$$

and that (by § 6)

$$D'_3(n) = \sigma_3(\tfrac{1}{2}n), \qquad D'(n) = \sigma(\tfrac{1}{2}n).$$

$$
\begin{array}{llllllll}
24\Sigma\sigma\sigma & = & 10\sigma_3(n) & & -12n\sigma(n) & +2\sigma(n) & & \\
24\Sigma\Delta\Delta & = & 2\sigma_3(n) & +\ 8D'_3(n) & & -2\sigma(n) & & +\ 4D'(n), \\
24\Sigma DD & = & & 40D'_3(n) & & & -24nD'(n) & +\ 8D'(n), \\
24\Sigma\Delta'\Delta' & = & 6\sigma_3(n) & -\ 6D'_3(n) & -\ 6n\sigma(n) & & +\ 6nD'(n), & \\
24\Sigma D'D' & = & & 10D'_3(n) & & & -\ 6nD'(n) & +\ 2D'(n), \\
24\Sigma\zeta\zeta & = - & 6\sigma_3(n) & +96D'_3(n) & +12n\sigma(n) & -6\sigma(n) & -48nD'(n) & +24D'(n), \\
24\Sigma\sigma\Delta & = & 6\sigma_3(n) & -16D'_3(n) & -\ 6n\sigma(n) & & +12nD'(n) & -\ 2D'(n), \\
24\Sigma\sigma D & = & 4\sigma_3(n) & +16D'_3(n) & -\ 6n\sigma(n) & +2\sigma(n) & -12nD'(n) & +\ 2D'(n), \\
24\Sigma\sigma\Delta' & = & 8\sigma_3(n) & -\ 8D'_3(n) & -\ 9n\sigma(n) & +\ \sigma(n) & +\ 6nD'(n) & -\ D'(n), \\
24\Sigma\sigma D' & = & 2\sigma_3(n) & +\ 8D'_3(n) & -\ 3n\sigma(n) & +\ \sigma(n) & -\ 6nD'(n) & +\ D'(n), \\
24\Sigma\sigma\zeta & = & 2\sigma_3(n) & -32D'_3(n) & & -2\sigma(n) & +24nD'(n) & -\ 4D'(n), \\
24\Sigma\Delta D & = & 4\sigma_3(n) & -24D'_3(n) & -\ 6n\sigma(n) & +2\sigma(n) & +12nD'(n) & -\ 6D'(n), \\
24\Sigma\Delta\Delta' & = & 4\sigma_3(n) & -\ 4D'_3(n) & -\ 3n\sigma(n) & -\ \sigma(n) & +\ 6nD'(n) & +\ D'(n), \\
24\Sigma\Delta D' & = & 2\sigma_3(n) & -12D'_3(n) & -\ 3n\sigma(n) & +\ \sigma(n) & +\ 6nD'(n) & -\ 3D'(n), \\
24\Sigma\Delta\zeta & = - & 2\sigma_3(n) & +32D'_3(n) & +\ 6n\sigma(n) & -4\sigma(n) & -12nD'(n) & +10D'(n), \\
24\Sigma D\Delta' & = & 4\sigma_3(n) & -\ 4D'_3(n) & -\ 6n\sigma(n) & +2\sigma(n) & & -\ 2D'(n), \\
24\Sigma DD' & = & & 20D'_3(n) & & & -12nD'(n) & +\ 4D'(n), \\
24\Sigma D\zeta & = & 4\sigma_3(n) & -64D'_3(n) & -\ 6n\sigma(n) & +2\sigma(n) & +36nD'(n) & -14D'(n), \\
24\Sigma\Delta'D' & = & 2\sigma_3(n) & -\ 2D'_3(n) & -\ 3n\sigma(n) & +\ \sigma(n) & & -\ D'(n), \\
24\Sigma\Delta'\zeta & = & & & 3n\sigma(n) & -3\sigma(n) & +\ 6nD'(n) & +\ 3D'(n), \\
24\Sigma D'\zeta & = & 2\sigma_3(n) & -32D'_3(n) & -\ 3n\sigma(n) & +\ \sigma(n) & +18nD'(n) & -\ 7D'(n),
\end{array}
$$

§ 14. Considering only the product-sums of the form Σff, we may deduce at once from the results on the last page the following formulæ:

$$12\Sigma\sigma\sigma = \quad 5\sigma_3(n) - 6n\sigma(n) + \sigma(n),$$
$$24\Sigma DD = \quad 5D_3(n) - 12nD(n) + 4D(n),$$
$$4\Sigma\Delta'\Delta' = \quad \Delta'_3(n) - n\Delta'(n),$$
$$12\Sigma D'D' = \quad 5D'_3(n) - 3nD'(n) + D'(n),$$
$$4\Sigma\zeta\zeta = -\zeta_3(n) + 2n\zeta(n) - \zeta(n).$$

These formulæ are all of the kind referred to on p. 105 of vol. XIV., viz. they serve to express $\sigma_3(n)$, $D_3(n)$, &c. in terms of $\sigma(n)$, $D(n)$, &c., respectively, *i.e.* in terms of functions in which the divisors themselves are involved in exactly the same way as their cubes are involved in the original functions.

The value of $\Sigma\Delta\Delta$ does not admit of being expressed in terms of $\Delta'_3(n)$ alone, or in terms of only one kind of function. Since $\zeta'(n) = \Delta(n)$, we have

$$\Sigma\Delta\Delta = \Sigma\Delta\zeta' = \Sigma\zeta'\zeta',$$

and it seems therefore most natural to select $\zeta'_3(n)$ as the second function by means of which to express the value of this product-sum. The resulting formula is easily seen to be

$$12\Sigma\Delta\Delta = 2\zeta_3'(n) - \Delta_3(n) - \Delta(n).$$

With reference to the system of formulæ on the last page it may be remarked that, since $D(n) = 2D'(n)$, it is evident that

$$\Sigma Df = 2\Sigma D'f,$$

and

$$\Sigma DD = 4\Sigma D'D'.$$

§ 15. Since $D'(r) = 0$, where r is uneven, it follows that

$$\begin{aligned}\Sigma\sigma D' &= \sigma(1)D'(n-1) + \sigma(2)D'(n-2)\ldots + \sigma(n-1)D'(1)\\ &= D(2)\,\sigma(n-2) + D(4)\,\sigma(n-4) + \ldots\\ &= \sigma(1)\,\sigma(n-2) + \sigma(2)\,\sigma(n-4) + \ldots,\end{aligned}$$

the last term being $\sigma(\tfrac{1}{2}n-1)\,\sigma(2)$ or $\sigma(\tfrac{1}{2}n-\tfrac{1}{2})\,\sigma(1)$ according as n is even or uneven.

Also

$$\begin{aligned}\Sigma D'D' &= D'(1)\,D'(n-1) + D'(2)\,D'(n-2)\ldots + D'(n-1)\,D'(1)\\ &= D'(2)\,D'(n-2) + D'(4)\,D'(n-4)\ldots + D'(n-2)\,D'(2).\end{aligned}$$

Thus $\Sigma D'D' = 0$ unless n is even, and in that case

$$\Sigma D'D' = \sigma(1)\,\sigma(\tfrac{1}{2}n-1) + \sigma(2)\,\sigma(\tfrac{1}{2}n-2) \ldots + \sigma(\tfrac{1}{2}n-1)\,\sigma(1).$$

The case of n uneven, § 16.

§ 16. If n be an uneven number, then

$$\Sigma DD = 0, \quad \Sigma DD' = 0, \quad \Sigma D'D' = 0.$$

Since $D_3'(n)$ and $D'(n)$ are equal to zero, when n is uneven, we may deduce at once, from § 13, the following system of formulæ, in which m denotes any uneven number, and

$$\Sigma fF = f(1)\,F(m-1) + f(2)\,F(m-2) \ldots + f(m-1)\,F(1).$$

$$\begin{aligned}
24\Sigma\sigma\sigma &= 10\sigma_3(m) - 12m\sigma(m) + 2\sigma(m),\\
24\Sigma\Delta\Delta &= 2\sigma_3(m) - 2\sigma(m),\\
24\Sigma\Delta'\Delta' &= 6\sigma_3(m) - 6m\sigma(m),\\
24\Sigma\zeta\zeta &= -6\sigma_3(m) + 12m\sigma(m) - 6\sigma(m),\\
24\Sigma\sigma\Delta &= 6\sigma_3(m) - 6m\sigma(m),\\
24\Sigma\sigma D &= 4\sigma_3(m) - 6m\sigma(m) + 2\sigma(m),\\
24\Sigma\sigma\Delta' &= 8\sigma_3(m) - 9m\sigma(m) + \sigma(m),\\
24\Sigma\sigma D' &= 2\sigma_3(m) - 3m\sigma(m) + \sigma(m),\\
24\Sigma\sigma\zeta &= 2\sigma_3(m) - 2\sigma(m),\\
24\Sigma\Delta D &= 4\sigma_3(m) - 6m\sigma(m) + 2\sigma(m),\\
24\Sigma\Delta\Delta' &= 4\sigma_3(m) - 3m\sigma(m) - \sigma(m),\\
24\Sigma\Delta D' &= 2\sigma_3(m) - 3m\sigma(m) + \sigma(m),\\
24\Sigma\Delta\zeta &= -2\sigma_3(m) + 6m\sigma(m) - 4\sigma(m),\\
24\Sigma D\Delta' &= 4\sigma_3(m) - 6m\sigma(m) + 2\sigma(m),\\
24\Sigma D\zeta &= 4\sigma_3(m) - 6m\sigma(m) + 2\sigma(m),\\
24\Sigma\Delta' D' &= 2\sigma_3(m) - 3m\sigma(m) + \sigma(m),\\
24\Sigma\Delta'\zeta &= 3m\sigma(m) - 3\sigma(m),\\
24\Sigma D'\zeta &= 2\sigma_3(m) - 3m\sigma(m) + \sigma(m).
\end{aligned}$$

In all the expressions which occur upon the right-hand side of these equations the sum of the coefficients is equal to zero. This is a condition which must obviously be satisfied, since each expression must reduce to zero when $m = 1$.

The case of n even, §§ 17–19.

§ 17. If n be uneven, the argument of one of the two functions, which occur in each term of ΣfF, is even, and that of the other is uneven; but, if n be even, ΣfF consists of two sets of terms of different kinds, all the arguments of the functions in the one set being even, and, in the other, uneven. Denoting these two sets of terms by $\Sigma_1 fF$ and $\Sigma_2 fF$, we have, n being supposed even,

$$\Sigma fF = \Sigma_1 fF + \Sigma_2 fF,$$

where

$$\Sigma_1 fF = f(1)\,F(n-1) + f(3)\,F(n-3) \ldots + f(n-1)\,F(1),$$

$$\Sigma_2 fF = f(2)\,F(n-2) + f(4)\,F(n-4) \ldots + f(n-4)\,F(2).$$

Now, from p. 103 of vol. XIV., n being uneven

$$\Sigma_1 \sigma\sigma = \sigma(1)\,\sigma(n-1) + \sigma(3)\,\sigma(n-3) \ldots + \sigma(n-1)\,\sigma(1)$$

$$= \Delta'_3(\tfrac{1}{2}n) = \tfrac{1}{8}\Delta_3'(n) = \tfrac{1}{8}\{\sigma_3(n) - D'_3(n)\},$$

and it is evident that

$$\Sigma_1 \sigma D' = 0,$$

and

$$\Sigma_1 D'D' = 0.$$

We may, therefore, in § 10, replace Σ by Σ_1, if we put

$$(ss) = \tfrac{1}{8}\{\sigma_3(n) - D'_3(n)\}, \quad (sd') = 0, \quad (d'd') = 0.$$

Thus the ten product-sums

$$\Sigma_1\sigma\sigma, \quad \Sigma_1\Delta\Delta, \quad \Sigma_1\Delta'\Delta', \quad \Sigma_1\zeta\zeta, \quad \Sigma_1\sigma\Delta,$$

$$\Sigma_1\sigma\Delta', \quad \Sigma_1\sigma\zeta, \quad \Sigma_1\Delta\Delta', \quad \Sigma_1\Delta\zeta, \quad \Sigma_1\Delta'\zeta,$$

are each equal to $\frac{1}{8}\{\sigma_3(n) - D_3'(n)\}$, and the remaining eleven product-sums $\Sigma_1 DD$, $\Sigma_1 D'D'$, $\Sigma_1\sigma D$, &c. are each equal to zero.

The equality of the ten expressions $\Sigma_1\sigma\sigma$, $\Sigma_1\Delta\Delta$, &c. is evident independently, since, when r is uneven,

$$\sigma(r) = \Delta(r) = \Delta'(r) = \zeta(r).$$

Since $D(r)=0$ and $D'(r)=0$, it is also evident that the product-sums which involve D or D' must vanish.

§ 18. Since

$$\Sigma_2\sigma\sigma = \Sigma\sigma\sigma - \Sigma_1\sigma\sigma,$$

$$\Sigma_2\sigma D' = \Sigma\sigma D',$$

$$\Sigma_2 D'D' = \Sigma D'D',$$

it is evident from the formulæ in § 10 that the value of $24\Sigma_2 fF$ may be derived from that of $24\Sigma fF$ by subtracting the value of $24\Sigma_1 fF$.

It was shown in the last section that

$$24\Sigma_1 fF = 3\sigma_3(n) - 3D'_3(n)$$

if neither f nor F is D or D', but otherwise $= 0$, and we thus deduce from § 13 the following formulæ, in which, for brevity, only the coefficients of the terms are written, so that, for example,

$$24\Sigma_2 \Delta\Delta = -\sigma_3(n) + 11D'_3(n) - 2\sigma(n) + 4D'(n).$$

	σ_3	D'_3	$n\sigma$	σ	nD'	D'
$24\Sigma_2\sigma\sigma =$	7	+ 3	− 12	+ 2		
$24\Sigma_2\Delta\Delta =$	− 1	+ 11		− 2		+ 4,
$24\Sigma_2 DD =$		40			− 24	+ 8,
$24\Sigma_2\Delta'\Delta' =$	3	− 3	− 6		+ 6	
$24\Sigma_2 D'D' =$		10			− 6	+ 2,
$24\Sigma_2\zeta\zeta =$	− 9	+ 99	+ 12	− 6	− 48	+ 24,
$24\Sigma_2\sigma\Delta =$	3	− 13	− 6		+ 12	− 2,
$24\Sigma_2\sigma D =$	4	+ 16	− 6	+ 2	− 12	+ 2,
$24\Sigma_2\sigma\Delta' =$	5	− 5	− 9	+ 1	+ 6	− 1,
$24\Sigma_2\sigma D' =$	2	+ 8	− 3	+ 1	− 6	+ 1,
$24\Sigma_2\sigma\zeta =$	− 1	− 29		− 2	+ 24	− 4,
$24\Sigma_2\Delta D =$	4	− 24	− 6	+ 2	+ 12	− 6,
$24\Sigma_2\Delta\Delta' =$	1	− 1	− 3	− 1	+ 6	+ 1,
$24\Sigma_2\Delta D' =$	2	− 12	− 3	+ 1	+ 6	− 3,
$24\Sigma_2\Delta\zeta =$	− 5	+ 35	+ 6	− 4	− 12	+ 10,
$24\Sigma_2 D\Delta' =$	4	− 4	− 6	+ 2		− 2,
$24\Sigma_2 DD' =$		20			− 12	+ 4,
$24\Sigma_2 D\zeta =$	4	− 64	− 6	+ 2	+ 36	− 14,
$24\Sigma_2\Delta' D' =$	2	− 2	− 3	+ 1		− 1,
$24\Sigma_2\Delta'\zeta =$	− 3	+ 3	+ 3	− 3	+ 6	+ 3,
$24\Sigma_2 D'\zeta =$	2	− 32	− 3	+ 1	+ 18	− 7.

§ 19. On pp. 107 and 108 of vol. XIV. it was shown that, if n be even,

$$-\zeta_3(n)=3\Delta(n)+4\{\Delta(1)\Delta(n-1)+9\Delta(2)\Delta(n-2)$$
$$+\Delta(3)\Delta(n-3)+9\Delta(4)\Delta(n-4)\ldots+\Delta(n-1)\Delta(1)\},$$

and
$$(2n-1)\zeta(n)+3\Delta(n)=$$
$$4\{\zeta(2)\zeta(n-2)+\zeta(4)\zeta(n-4)\ldots+\zeta(n-2)\zeta(2)\}$$
$$-36\{\Delta(2)\Delta(n-2)+\Delta(4)\Delta(n-4)\ldots+\Delta(n-2)\Delta(2)\}.$$

These formulæ, which in the notation of this paper may be written

$$4\Sigma_1\Delta\Delta+36\Sigma_2\Delta\Delta=-\zeta_3(n)-3\Delta(n),$$
$$4\Sigma_2\zeta\zeta\;-36\Sigma_2\Delta\Delta=\quad(2n-1)\zeta(n)+3\Delta(n),$$

afford verifications of some of the results given in the last two sections; for, by §§ 17 and 18,

$$8\Sigma_1\Delta\Delta+72\Sigma_2\Delta\Delta=\sigma_3(n)-D'_3(n)$$
$$-3\sigma_3(n)+33D'_3(n)-6\sigma(n)+12D'(n)$$
$$=-2\zeta_3(n)-6\Delta(n),$$

and, writing the coefficients only,

$$24\Sigma_2\zeta\zeta-216\Sigma_2\Delta\Delta=-9+99+12-\;6-48+24$$
$$+9-99\qquad+18\qquad-36$$
$$=12n\sigma(n)+12\sigma(n)-48nD'(n)-12D'(n)$$
$$=12n\zeta(n)+12\Delta'(n),$$

which agree with the results quoted above.

Series having $\sigma_s(n)$, $\Delta_s(n)$, &c., as coefficients, §§ 20–21.

§ 20. The simplest algebraical forms of the series

$$\Sigma_1^\infty f_s(n)\,x^n,$$

in which the coefficient of x^n is the arithmetical function $f_s(n)$

of the divisors of n, are given by the following equations:

$$\Sigma_1^\infty \ \sigma_s(n)\, x^n = \Sigma_1^\infty \ \frac{n^s x^n}{1-x^n},$$

$$\Sigma_1^\infty \Delta_s(n)\, x^n = \Sigma_1^\infty \ \frac{(2n-1)^s x^{2n-1}}{1-x^{2n-1}},$$

$$\Sigma_1^\infty D_s(n)\, x^n = \Sigma_1^\infty \ \frac{(2n)^s x^{2n}}{1-x^{2n}},$$

$$\Sigma_1^\infty \Delta'_s(n)\, x^n = \Sigma_1^\infty \ \frac{n^s x^n}{1-x^{2n}},$$

$$\Sigma_1^\infty D'_s(n)\, x^n = \Sigma_1^\infty \ \frac{n^s x^{2n}}{1-x^{2n}},$$

$$\Sigma_1^\infty \ \zeta_s(n)\, x^n = \Sigma_1^\infty (-)^{n-1} \frac{n^s x^n}{1-x^n},$$

$$\Sigma_1^\infty \ \zeta'_s(n)\, x^n = \Sigma_1^\infty \ \frac{n^s x^n}{1+x^n};$$

from which we may derive by differentiation the equations

$$\Sigma_1^\infty \ n\sigma_s(n)\, x^n = \Sigma_1^\infty \ \frac{n^{s+1} x^n}{(1-x^n)^2},$$

$$\Sigma_1^\infty \ n\Delta_s(n)\, x^n = \Sigma_1^\infty \ \frac{(2n-1)^{s+1} x^{2n-1}}{(1-x^{2n-1})^2},$$

$$\Sigma_1^\infty \ nD_s(n)\, x^n = \Sigma_1^\infty \ \frac{(2n)^{s+1} x^{2n}}{(1-x^{2n})^2},$$

$$\Sigma_1^\infty n\Delta'_s(n)\, x^n = \Sigma_1^\infty \ \frac{n^{s+1} x^n (1+x^{2n})}{(1-x^{2n})^2},$$

$$\Sigma_1^\infty nD'_s(n)\, x^n = \Sigma_1^\infty \ \frac{2n^{s+1} x^{2n}}{(1-x^{2n})^2},$$

$$\Sigma_1^\infty \ n\zeta_s(n)\, x^n = \Sigma_1^\infty (-)^{n-1} \frac{n^{s+1} x^n}{(1-x^n)},$$

$$\Sigma_1^\infty \ n\zeta'_s(n)\, x^n = \Sigma_1^\infty \ \frac{n^{s+1} x^n}{(1+x^n)^2}.$$

§ 21. The formulæ in § 13 afford expressions for the squares of the quantities in the first system of equations, when $s = 1$, and for the products of any pair of them. Thus, for example,

$$24\Sigma\Delta\Delta = 2\sigma_3(n) + 8D'_3(n) - 2\sigma(n) + 4D'(n)$$
$$= 4\zeta_3'(n) - 2\Delta_3(n) - 2\Delta(n), \text{ by § 14;}$$

and, therefore,

$$12\{\Sigma_1^\infty \Delta(n)x^n\}^2 = \Sigma_1^\infty \{2\zeta_3'(n) - \Delta_3(n) - \Delta(n)\}x^n,$$

or, writing the series in their algebraical forms,

$$12\left\{\Sigma_1^\infty \frac{(2n-1)x^{2n-1}}{1-x^{2n-1}}\right\}^2$$
$$= 2\Sigma_1^\infty \frac{n^3x^3}{1+x^n} - \Sigma_1^\infty \frac{(2n-1)^3x^{2n-1}}{1-x^{2n-1}} - \Sigma_1^\infty \frac{(2n-1)x^{2n-1}}{1-x^{2n-1}}.$$

Similarly

$$24\Sigma D\Delta' = 4\sigma_3(n) - 4D'_3(n) - 6n\sigma(n) + 2\sigma(n) - 2D'(n)$$
$$= 4\Delta_3'(n) - 6n\sigma(n) + 2\Delta'(n),$$

whence

$$12\{\Sigma_1^\infty D(n)x^n\} \times \{\Sigma_1^\infty \Delta'(n)x^n\} = \Sigma_1^\infty\{2\Delta'_3(n) - 3n\sigma(n) + \Delta'(n)\}x^n,$$

or $$12\left\{\Sigma_1^\infty \frac{2nx^{2n}}{1-x^{2n}}\right\} \times \left\{\Sigma_1^\infty \frac{nx^n}{1-x^{2n}}\right\}$$
$$= 2\Sigma_1^\infty \frac{n^3x^n}{1-x^{2n}} - 3\Sigma_1^\infty \frac{n^2x^n}{(1-x^n)^2} + \Sigma_1^\infty \frac{nx^n}{1-x^{2n}}.$$

Values of elliptic-function series, § 22.

§ 22. The values of the elliptic-function q-series, which may thus be connected with the formulæ in § 13, are as follows:

$$\Sigma_1^\infty \sigma(n)q^n = \tfrac{1}{24} - \frac{KE + 4KI + KG}{6\pi^2},$$

$$\Sigma_1^\infty \Delta(n)q^n = -\tfrac{1}{24} + \frac{KE - 2KI + KG}{6\pi^2},$$

$$\Sigma_1^\infty D(n)q^n = \tfrac{1}{12} - \frac{KE + KI + KG}{3\pi^2},$$

$$\Sigma_1^\infty \Delta'(n)\, q^n = \qquad -\frac{KI}{2\pi^2},$$

$$\Sigma_1^\infty D'(n)\, q^n = \quad \tfrac{1}{24} - \frac{KE+KI+KG}{6\pi^2},$$

$$\Sigma_1^\infty \; \zeta(n)\, q^n = -\tfrac{1}{8} + \frac{KE+KG}{2\pi^2},$$

$$\Sigma_1^\infty \; \zeta'(n)\, q^n = -\tfrac{1}{24} + \frac{KE-2KI+KG}{6\pi^2}.$$

[The value of the equal series $\Sigma_1^\infty \Delta(n)\, q^n$ and $\Sigma_1^\infty \zeta'(n)\, q^n$ may be expressed also in the form

$$-\tfrac{1}{24} + \frac{(1+k^2)\, K^2}{6\pi^2}$$

in which only K^2 and k^2 are involved.]

$$\Sigma_1^\infty \; n\sigma(n)\, q^n = -\frac{2K^2(2IG-GE+2EI)}{3\pi^4},$$

$$\Sigma_1^\infty \; n\Delta(n)\, q^n = -\frac{2K^2(IG-2GE+EI)}{3\pi^4},$$

$$\Sigma_1^\infty \; nD(n)\, q^n = -\frac{2K^2(IG+GE+EI)}{3\pi^4},$$

$$\Sigma_1^\infty n\Delta'(n)\, q^n = -\frac{K^2(IG-GE+EI)}{\pi^4},$$

$$\Sigma_1^\infty nD'(n)\, q^n = -\frac{K^2(IG+GE+EI)}{3\pi^4},$$

$$\Sigma_1^\infty \; n\zeta(n)\, q^n = \quad \frac{2K^2GE}{\pi^4},$$

$$\Sigma_1^\infty \; n\zeta'(n)\, q^n = -\frac{2K^2(IG-2GE+EI)}{3\pi^4};$$

$$\Sigma_1^\infty \sigma_3(n)q^n = -\tfrac{1}{240} + \frac{(1+14k^2+k^4)K^4}{15\pi^4},$$

$$\Sigma_1^\infty \Delta_3(n)q^n = \tfrac{7}{240} - \frac{(7-22k^2+7k^4)K^4}{15\pi^4},$$

$$\Sigma_1^\infty D_3(n)q^n = -\tfrac{1}{30} + \frac{8(1-k^2k'^2)K^4}{15\pi^4},$$

$$\Sigma_1^\infty \Delta'_3(n)q^n = \frac{k^2K^4}{\pi^4},$$

$$\Sigma_1^\infty D'_3(n)q^n = -\tfrac{1}{240} + \frac{(1-k^2k'^2)K^4}{15\pi^4},$$

$$\Sigma_1^\infty \zeta_3(n)q^n = \tfrac{1}{16} - \frac{k'^4K^4}{\pi^4},$$

$$\Sigma_1^\infty \zeta_3'(n)q^n = \tfrac{1}{240} - \frac{(1-16k^2+k^4)K^4}{15\pi^4}.$$

ON THE VIBRATIONS OF A GAS INCLUDED BETWEEN TWO CONCENTRIC SPHERICAL SURFACES, WITH SPECIAL REFERENCE TO THE CASE WHEN THE BOUNDING RADII ARE NEARLY EQUAL.*

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THE following seems a concise and satisfactory method. Referring to page 204 of Lord Rayleigh's *Sound*, vol. II., we see that the velocity potential may be expanded in a series of terms involving spherical harmonics of different degrees. Considering ψ_n, which contains a spherical harmonic of the n^{th} degree, as a type, we have to determine ψ_n as a function of r from the equation

$$r^2\frac{d^2\psi_n}{dr^2} + 2r\frac{d\psi_n}{dr} - n(n+1)\psi_n + k^2r^2\psi_n = 0 \quad \ldots(1).$$

This may be written in the form

$$\frac{d^2}{dr^2}(r^{\frac{1}{2}}\psi_n) + \frac{1}{r}\frac{d}{dr}(r^{\frac{1}{2}}\psi_n) + \left\{k^2 - \frac{(n+\frac{1}{2})^2}{r^2}\right\}r^{\frac{1}{2}}\psi_n = 0 \ldots (2).$$

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* See Rayleigh's *Sound*, vol. II., §§ 336 and 323.

Supposing n an integer, and so $n+\frac{1}{2}$ not an integer, the solution is

$$r^{\frac{1}{2}}\psi_n = AJ_{n+\frac{1}{2}}(kr) + BJ_{-(n+\frac{1}{2})}(kr) \ldots\ldots\ldots\ldots(3),$$

where A and B are arbitrary constants and J denotes as usual a Bessel's function.

[If the gas extended to the centre we should have $B=0$].

Writing in now the factor depending on the time, of which the above equation assumes the period to be $\frac{2\pi}{ck}$, and the surface harmonic of the n^{th} degree, which we shall denote by Y_n, we have as a type of the complete solution

$$\psi_n = Y_n \cos(kct-\alpha)\, r^{-\frac{1}{2}} \{AJ_{n+\frac{1}{2}}(kr) + BJ_{-(n+\frac{1}{2})}(kr)\} \ldots(4),$$

where c is the velocity of sound in the gas, and α is given by the data as to the initial time.

Suppose now a spherical shell bounded by rigid surfaces $r=a$, and $r=a'$ inside which the gas is vibrating.

The surface conditions are $\frac{d\psi_n}{dr}=0$ when $r=a$ and when $r=a'$. We have thus to determine the value of k, and of $B:A$, from

$$\left.\begin{aligned} &A\,[kaJ'_{n+\frac{1}{2}}(ka) - \tfrac{1}{2}J_{n+\frac{1}{2}}(ka)] \\ &\qquad + B\,[kaJ'_{-(n+\frac{1}{2})}(ka) - \tfrac{1}{2}J_{-(n+\frac{1}{2})}(ka)] = 0 \\ &A\,[ka'J'_{n+\frac{1}{2}}(ka') - \tfrac{1}{2}J_{n+\frac{1}{2}}(ka')] \\ &\qquad + B\,[ka'J'_{-(n+\frac{1}{2})}(ka') - \tfrac{1}{2}J_{-(n+\frac{1}{2})}(ka')] = 0 \end{aligned}\right\}\ldots(5),$$

where we denote $\frac{dJ(kr)}{d.kr}$ by $J'(kr)$.

The values of k, and so the periods of free vibration corresponding to the n^{th} surface harmonic, are given by eliminating $B:A$ from the above equations.

The result is

$$\begin{aligned} &ka'J'_{n+\frac{1}{2}}(ka')\,J_{-(n+\frac{1}{2})}(ka) - kaJ'_{n+\frac{1}{2}}(ka)\,J_{-(n+\frac{1}{2})}(ka') \\ &+ kaJ'_{-(n+\frac{1}{2})}(ka)\,J_{n+\frac{1}{2}}(ka') - ka'J'_{-(n+\frac{1}{2})}(ka')\,J_{n+\frac{1}{2}}(ka) \\ &+ k^2aa'\,[J'_{n+\frac{1}{2}}(ka)\,J'_{-(n+\frac{1}{2})}(ka') - J'_{-(n+\frac{1}{2})}(ka)\,J'_{n+\frac{1}{2}}(ka')] \\ &+ \tfrac{1}{4}\,[J_{n+\frac{1}{2}}(ka)\,J_{-(n+\frac{1}{2})}(ka') - J_{n+\frac{1}{2}}(ka')\,J_{-(n+\frac{1}{2})}(ka)] = 0 \ldots(6). \end{aligned}$$

In the general case, when a and a' are not nearly equal, this would be a cumbrous equation; but we shall consider specially the case when $a'-a$ is small.

We have

$$ka'J'_{n+\frac{1}{2}}(ka') = kaJ'_{n+\frac{1}{2}}(ka) + k(a'-a)\frac{d}{d.ka}\{kaJ'_{n+\frac{1}{2}}(ka)\}$$

+ terms in $(a'-a)^2$, &c., which we shall here neglect;

but

$$\frac{d}{d.ka}\{kaJ'_{n+\frac{1}{2}}(ka)\} = -kaJ_{n+\frac{1}{2}}(ka)\left\{1-\frac{(n+\frac{1}{2})^2}{k^2a^2}\right\};$$

therefore

$$ka'J'_{n+\frac{1}{2}}(ka') = kaJ'_{n+\frac{1}{2}}(ka) - k(a'-a)\,kaJ_{n+\frac{1}{2}}(ka)\left\{1-\frac{(n+\frac{1}{2})^2}{k^2a^2}\right\},$$

so

$$ka'J'_{-(n+\frac{1}{2})}(ka') = kaJ'_{-(n+\frac{1}{2})}(ka)$$
$$-k(a'-a)\,kaJ_{-(n+\frac{1}{2})}(ka)\left\{1-\frac{(n+\frac{1}{2})^2}{k^2a^2}\right\}.$$

Using these values in (6) the first two lines cut out, and the last two reduce to the product of two factors which we shall denote by E and F.

The equation for k is thus $E.F=0$.

$$F \text{ is} \equiv k(a'-a)\left[k^2a^2\left\{1-\frac{(n+\frac{1}{2})^2}{k^2a^2}\right\}+\frac{1}{4}\right] \ldots\ldots (7),$$

i.e.

$$\equiv k(a'-a)\left[k^2a^2 - n(n+1)\right].$$

The value of E is

$$\equiv J'_{n+\frac{1}{2}}(ka)\,J_{-(n+\frac{1}{2})}(ka) - J'_{-(n+\frac{1}{2})}(ka)\,J_{n+\frac{1}{2}}(ka) \ldots (8).$$

It is, however, easy to shew that $E=\dfrac{C}{ka}$, where C is a numerical quantity wholly independent of ka.

This is a special case of the following relation.

If $J_m(x)$, $Y_m(x)$ denote the two solutions of the general form of the Bessel's equation

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+u\left(1-\frac{m^2}{x^2}\right)=0,$$

then is

$$J'_m(x)\,Y_m(x) - Y'_m(x)\,J_m(x) = \frac{C}{x} \ldots\ldots\ldots\ldots (9),$$

where C is wholly independent of x. This follows at once if we write the Bessel's equation in the form

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) = - xu\left(1 - \frac{m^2}{x^2}\right),$$

which shews that

$$\frac{d}{dx}xJ'_m(x) = - xJ_m(x)\left(1 - \frac{m^2}{x^2}\right),$$

and
$$\frac{d}{dx}xY'_m(x) = - xY_m(x)\left(1 - \frac{m^2}{x^2}\right),$$

Thus

$$\frac{dC}{dx} \text{ being } = Y_m(x)\frac{d}{dx}xJ'_m(x) - J_m(x)\frac{d}{dx}xY'_m(x),$$

$$\text{is } = J_m(x)\,Y_m(x)\,(0);$$

therefore $\dfrac{dC}{dx} = 0$, or C is independent of x.

If now
$$m = n + \tfrac{1}{2},$$

$$Y_m(x) \text{ is } \equiv J_{-(n+\frac{1}{2})}(x);$$

and, writing $x \equiv ka$, we have

$$E = \frac{C}{ka},$$

and
$$E.F = C\frac{(a' - a)}{a}\left[k^2a^2 - n(n + 1)\right],$$

and the roots of the equation $E.F = 0$ are thus given by

$$k^2a^2 = n(n + 1) \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

This is the result given by Lord Rayleigh.

The advantage of the above method is that up to (6) it is independent of any assumption as to the thickness of the shell, so that the change in the pitch when terms in $(a' - a)^2$ are retained can be obtained by a closer approximation in (6) with comparative ease. Moreover, if the gas be bounded by membranes which are subjected to radial displacements, which are periodic functions of the time, we have only to express these displacements in a series of spherical surface harmonics and introduce the corresponding velocities in the right side of (5). We thus determine A and B and the corresponding "forced" vibration in the gas. This obviously becomes infinite if the period of the forced vibration coincides with that of one of the free vibrations as given by (6).

ON TRIANGLES OF MAXIMUM AND MINIMUM AREA INSCRIBED IN A PLANE CUBIC.

By *R. A. Roberts, M.A.*

Before proceeding to the analytical investigation of the conditions which serve to determine triangles of maximum or minimum area inscribed in a cubic, it is worth while noticing in what cases such a problem is geometrically possible. For triangles of maximum area it is evident that the curve must possess an oval on which the three vertices of the triangle will lie. In this case the curve may have a node, the triangle then being confined to the loop, but cannot possibly have a cusp. For triangles of minimum area the curve must have three distinct infinite branches. Each vertex of the triangle may lie on one of these branches, but if the curve have an oval besides, one vertex may lie on the oval and the two others on two different infinite branches. In the case of the minimun triangles also, the curve may have a node, as the oval may evidently reduce to such a point.

Since for a triangle of maximum or minimum area inscribed in a curve, the tangents at the vertices must be parallel to the opposite sides; the problem is to inscribe a triangle in the curve so that the tangents at the vertices meet the opposite sides in points lying on a given line, viz. the line at infinity.

When this is the case for a cubic, the triangle must meet the curve again in three points lying on a line. Now by the theory of the cubic there are three distinct systems of such triangles for the non-singular curve, one for the nodal curve and none for the curve with a cusp.

Let U be one of the cubics which has the given non-singular cubic V for its Hessian, then if x, y, z form a triangle inscribed in V, so that the points where the sides meet the curve again lie on the line u, it is known that we may write

$$U \equiv \alpha x^3 + \beta y^3 + \gamma z^3 + \delta u^3,$$

$$V \equiv \frac{yzu}{\alpha} + \frac{zux}{\beta} + \frac{uxy}{\gamma} + \frac{xyz}{\delta},$$

where $$x + y + z + u = 0.$$

It is evident then that the points where the tangents at the vertices meet the opposite sides lie on the line

$$\alpha x + \beta y + \gamma z \equiv P = 0;$$

and what we want is to find u when we are given this line; for the triangle xyz will evidently be uniquely determined when u is given, as the vertices are the points corresponding to those where u meets the curve.

Now if C be the Cayleyan of U it is easily shown that $C \equiv \Sigma (\lambda - \mu)(\lambda - \nu)(\lambda - \rho)\beta\gamma\delta$, where $\lambda x + \mu y + \nu z + \rho u = 0$, represents an arbitrary line. Hence, since $S = \alpha\beta\gamma\delta$, we have for the line u

$$CU - Su^3 = \alpha x^3 + \beta y^3 + \gamma z^3.$$

But the polar point of u with regard to C is

$$\lambda + \mu + \nu - 3\rho \equiv L = 0.$$

Operating then on $CU - Su^3$ with the square of L we get the required line $P \equiv \alpha x + \beta y + \gamma z = 0$.

Let us now write U in the canonical form

$$U \equiv x^3 + y^3 + z^3 + 6mxyz,$$

then the line passing through the points where the tangents at the vertices of the triangle meet the opposite sides is

$$\left(C_1 \frac{d}{dx} + C_2 \frac{d}{dy} + C_3 \frac{d}{dz}\right)^2 (CU - Su^3) = 0,$$

which, after dividing by C, becomes

$$x(C_1^2 + 2m C_2 C_3) + y(C_1^2 + 2m C_3 C_1) + z(C_3^2 + 2m C_1 C_2) - SCu = 0,$$

where C_1, C_2, C_3 are the three differentials of C.

Hence since P involves the coordinates of u in the fourth degree, it follows that there are sixteen positions of u when P is given. Also there are three cubics such as U; hence for the non-singular cubic we have forty-eight triangles altogether. For the nodal cubic the equation of P becomes divisible by the node, and the coordinates are of the third degree, and since there is then but one cubic of which V is the Hessian, we have nine triangles in this case.

ON CERTAIN CURVES OF THE SIXTH ORDER.

By *R. A. Roberts, M.A.*

It is known that the integral $\int \frac{dx}{(X)^{\frac{2}{3}}}$, where

$$X = (x-a)(x-b)(x-c),$$

depends upon an elliptic integral of the first kind, and we can easily shew independently that the equation

$$\frac{dx_1}{(X_1)^{\frac{2}{3}}} + \frac{dx_2}{(X_2)^{\frac{2}{3}}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

has an algebraic integral. Suppose we have

$$X + \lambda\phi(x) \equiv (px+q)^3 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

identically, where $\phi(x) = (x-x_1)(x-x_2)(x-x_3)$. Then considering x_1, x_2, x_3 as functions of p, q, we have by differentiation

$$\lambda\phi'(x_1)\,dx_1 = 3(px_1+q)^2(x_1 dp + dq),$$

or since, from (2), $px_1 + q = (X_1)^{\frac{1}{3}}$,

we get $$\frac{\lambda dx_1}{(X_1)^{\frac{2}{3}}} = \frac{3(x_1 dp + dq)}{\phi'(x_1)} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and similar relations for x_2, x_3. Hence we obtain

$$\sum\nolimits_3^1 \frac{dx}{(X)^{\frac{2}{3}}} = 0,$$

which coincides with (1) if we suppose x_3 to be constant. The algebraic relation may then be found from (2) and is evidently capable of being written in the form

$$(b-c)\sqrt[3]{\phi(a)} + (c-a)\sqrt[3]{\phi(b)} + (a-b)\sqrt[3]{\phi(c)} = 0 \ldots\ldots (4).$$

Now suppose $x_1 = x + iy$, $x_2 = x - iy$, $x_3 = k$, where x, y are rectangular Cartesian coordinates and k is a constant. Then (1) is the differential equation of a system of curves which from (4) we see are of the sixth order. Since, if we clear (4) of radicals we get an equation in k of the third degree, it follows that three curves of the system pass through a point.

For these three curves $\frac{dx_1}{dx_2}$ assumes the values

$$-\left(\frac{X_1}{X_2}\right)^{\frac{2}{3}}, -\theta\left(\frac{X_1}{X_2}\right)^{\frac{2}{3}}, -\theta^2\left(\frac{X_1}{X_2}\right)^{\frac{2}{3}},$$

respectively, where θ is a cube root of unity; and since

$$\frac{dx_1}{dx_2}=\frac{dx+idy}{dx-idy}=e^{2i\psi},$$

where ψ is the angle which the tangent to the curve makes with the axis of x, we hence infer that the three curves of the system which pass through a point cut each other at angles of 60°.

Let A, B, C, P be points on the axis of x corresponding to the values a, b, c, k respectively, and let ρ_1, ρ_2, ρ_3 be the distances of an arbitrary point from A, B, C, then, since

$$(a-x_1)(a-x_2)=\rho_1^2, \quad (b-x_1)(b-x_2)=\&c.,$$

we have from (4) the equation of the curve in the form

$$BC\sqrt[3]{(PA\rho_1^2)}+CA\sqrt[3]{(PB\rho_2^2)}+AB\sqrt[3]{(PC\rho_3^2)}=0\ldots.(5).$$

If we take the envelope of this equation (5) with regard to k, which is the same as supposing the point P to move along the axis of x, we get

$$BC\rho_1\pm CA\rho_2\pm AB\rho_3=0,$$

which being cleared of radicals gives $y^2=0$. But the curve is evidently symmetrical with regard to the axis of x. Hence it is very easy to see that the point $y=0$, $x=k$ is a double point on the curve. Also when a curve is written in the form $U^{\frac{1}{3}}+V^{\frac{1}{3}}+W^{\frac{1}{3}}=0$, the points determined by the equations $U=V=W$ are nodes. Thus we see that the two points determined by

$$BC^3\,PA\,\rho_1^2=CA^3\,PB\,\rho_2^2=AB^3\,PC\rho_3^2\ldots\ldots.(6)$$

are also nodes of the curves. The locus of these points for different positions of P is

$$(BC)^{-2}\rho_1^{-2}+(CA)^{-2}\rho_2^{-2}+(AB)^{-2}\rho_3^{-2}=0\ldots\ldots(7),$$

which is evidently altogether imaginary, from which we infer that the two points given by (6) are imaginary.

In equation (5) the points A, B, C lie on a line; but this restriction is removed by inverting the whole system from an

arbitrary point. The equation (5) remains unchanged, except that P is now supposed to move on the circle circumscribing the triangle A, B, C. These more general curves are then of the twelfth class, as the circular points are triple points, and we have seen that there are three finite double points. There are eighteen inflexional tangents, of which three pass through one circular point, and three through the other, these tangents giving rise to the foci A, B, C. We thus see that there are no other foci but these points and their anti-points, as the three inflexional tangents through I are the complete number which can be drawn through that point. Clearing the equation (5) of radicals we get

$$S^3 - 27\phi(k)\,\rho_1^2\rho_2^2\rho_3^2 = 0 \qquad (8),$$

where

$$S \equiv (3k - a - b - c)(x^2 + y^2) - 2\{k(a + b + c) - ab - bc - ca\}x + k(ab + bc + ca) - 3abc = 0 \qquad (9),$$

from which we see that the product of the distances of any point of the curve from the three foci is in a constant ratio to the cube of the tangent from the same point to a director circle S. These circles S, we see from (9), evidently pass through two points which remain fixed for every curve of the system. Again, from (8) we see that two curves of the system cannot intersect in more than six real points, as all the real points of intersection must evidently lie on the circle

$$\frac{S_1}{\sqrt[3]{\phi(k_1)}} - \frac{S_2}{\sqrt[3]{\phi(k_2)}} = 0 \qquad (10).$$

The more general curve which we have obtained above by inversion is evidently its own inverse with regard to the circle circumscribing the triangle ABC, and there is therefore a system of circles having double contact with the curve at a pair of inverse points. These circles being orthogonal to the circumscribing circle have an equation of the form

$$\lambda\rho_1^2 + \mu\rho_2^2 + \nu\rho_3^2 = 0 \qquad (11).$$

Expressing then that the circle (11) touches the curve (5), we get

$$\left(\frac{\lambda}{l}\right)^{-\frac{1}{2}} + \left(\frac{\mu}{m}\right)^{-\frac{1}{2}} + \left(\frac{\nu}{n}\right)^{-\frac{1}{2}} = 0 \qquad (12),$$

where $l = BC^3\,PA$, $m = CA^3\,PB$, $n = AB^3\,PC$.

Now λ, μ, ν are the areal coordinates of the centre of the circle (11) with regard to the triangle ABC. Thus we see that the centre of the circle lies on a quartic of which the three points A, B, C are cusps. Also from the values of l, m, n it is easily seen that the quartic (12) touches the circumscribing circle at the point P.

Hence, finally, we see that the curve may be generated as the envelope of a circle which cuts orthogonally a circle J and has its centre on a quartic having three cusps on J and touching J elsewhere.

Since in this mode of generation (see Casey's *Memoir on Bicircular Quartics*), the angle between the circles of the system (11) depends upon the anharmonic ratio in which the line joining their centres is divided by the circle J, we see that the common tangents of two curves of the system (12) are divided in a constant anharmonic ratio by the circle through the cusps.

I now proceed to shew that the curve can be generated as the envelope of a variable circle in three distinct ways. In order to prove this I obtain an integral of the equation (1) in a form different from (4). We may consider the equation (1) as resulting from the elimination of z_1, z_2 between the three equations

$$\frac{dx_1}{(X_1)^{\frac{2}{3}}} + \frac{dz_1}{(Z_1)^{\frac{2}{3}}} = 0 \quad\text{.................. (13)},$$

$$\frac{dx_2}{(X_2)^{\frac{2}{3}}} + \frac{dz_2}{(Z_2)^{\frac{2}{3}}} = 0 \quad\text{.................. (14)},$$

$$\frac{dz_1}{(Z_1)^{\frac{2}{3}}} + \frac{dz_2}{(Z_2)^{\frac{2}{3}}} = 0 \quad\text{.................. (15)}.$$

Now, it is evident that we shall obtain a general integral of (1) by taking a general integral of (15) and special integrals of (13) and (14), as for instance, those obtained by taking $k=a$ and b respectively in (4). We have thus for (13)

$$(c-a)^2 (b-x_1)(b-z_1) = (a-b)^2 (c-x_1)(c-z_1),$$

which we may write in the equivalent forms

$$\left.\begin{aligned} a-z_1 &= \lambda\,(a-b)(c-a)(a-x_1) \\ b-z_1 &= \lambda\,(a-b)^2 (c-x_1) \\ c-z_1 &= \lambda\,(c-a)^2 (b-x_1) \end{aligned}\right\} \quad\text{....... (16)}.$$

Similarly, by taking $k=b$, we have for (14)

$$\left.\begin{aligned} a-z_2 &= \lambda'(a-b)^2(c-x_2) \\ b-z_2 &= \lambda'(a-b)\ (b-c)\,(b-x_2) \\ c-z_2 &= \lambda'(b-c)^2(a-x_2) \end{aligned}\right\} \quad \ldots\ldots\ldots (17).$$

Substituting then the values of z_1, z_2 from (16) and (17) in the general integral of (15), we get

$$\sqrt[3]{l\rho_1^{\,2}} + \sqrt[3]{m\rho_2^{\,2}} + \sqrt[3]{n\rho_3^{\,2}} = 0 \quad \ldots\ldots\ldots\ldots\ldots (18),$$

where $$\left.\begin{aligned} &\rho_1^{\,2} = (a-x_1)(c-x_2),\ \rho_2^{\,2} = (c-x_1)(b-x_2) \\ &\rho_3^{\,2} = (b-x_1)(a-x_2) \end{aligned}\right\} \ldots (19),$$

and $$\left.\begin{aligned} &l = (b-c)^2(a-k),\ m = (c-a)^2(b-k) \\ &n = (b-c)(c-a)(c-k) \end{aligned}\right\} \ldots (20).$$

Thus we see that the equation of the curve is of exactly the same form as (5) when it is referred to three properly selected antipoints A', B', C' of A, B, C. We can also verify when the curve is written in the form (18), that it is the inverse of a curve of the form (5); for in this case, from Ptolemy's theorem, l, m, n should satisfy the equation

$$\frac{l}{B'C'^2} + \frac{m}{C'A'^2} + \frac{n}{A'B'^2} = 0 \quad \ldots\ldots\ldots\ldots (21);$$

but it is easy to see that

$$B'C'^2 = (a-b)(b-c),\ C'A'^2 = (a-b)(c-a),\ A'B'^2 = (b-c)(c-a);$$

hence, putting in for l, m, n from (20), we see that (21) is satisfied identically.

The third form of the equation of the curve is obtained by interchanging x_1 and x_2 in the values of ρ_1, ρ_2, ρ_3 in (19); and it is evident that no other distinct forms can be obtained by taking other special integrals of (13) and (14). Hence, by inversion, we see that the general curve can be generated in three distinct ways as the envelope of a variable circle which cuts orthogonally a fixed circle J, and has its centre on a tricuspidal quartic.

It can now be easily shown, that the equation (7) represents the product of two of the J circles, which are therefore always altogether imaginary. We see also that the three J circles are coaxal, being orthogonal to all the circles of the system S (9).

If we seek the orthogonal trajectory of the system of curves represented by the equation (1), we shall find

$$\frac{dx_1}{(X_1)^{\frac{2}{3}}} - \frac{dx_2}{(X_2)^{\frac{2}{3}}} = 0 \text{ .... } (22).$$

Now, in order to find the algebraic integral of this equation, we may consider it as resulting from the two equations

$$\frac{dx_1}{(X_1)^{\frac{2}{3}}} + \frac{dz_1}{(Z_1)^{\frac{2}{3}}} = 0, \quad \frac{dx_2}{(X_2)^{\frac{2}{3}}} + \frac{dz_2}{(Z_2)^{\frac{2}{3}}} = 0 \text{ ......} (23).$$

Taking then a special integral of one of these equations (23), as for instance, that corresponding to $k = c$ in (4), and eliminating z, we get

$$\sqrt[3]{(l\rho_1^2)} + \sqrt[3]{(m\rho_2^2)} + \sqrt[3]{(n\rho_3^2)} = 0 \text{ .... } (24),$$

where
$$l = (b-c)^2 (c-a) \; (a-k),$$
$$m = (b-c) \; (c-a)^2 (b-k),$$
$$n = (a-b)^3 (c-k) \text{ .... } (25),$$

and ρ_1, ρ_2 are now the distances of a point from the antipoints of A, B. We can shew then that this curve is of the same nature as the given one; for in order that this should be the case, l, m, n ought to satisfy the relation

$$\frac{l}{CB'^2} + \frac{m}{CA'^2} + \frac{n}{A'B'^2} = 0 \text{............} (26),$$

but
$$CA'^2 = CB'^2 = (c-a)(c-b),$$
and
$$A'B'^2 = -(a-b)^2,$$

hence, putting in for l, m, n from (25), we see that (26) is satisfied.

We see thus that there are two distinct systems of curves when the three foci are given. Two curves of the same system cut each other at angles of 60°, and two curves of different systems at right angles or angles of 30°. We have seen that two of the nodes of the first system are always imaginary, but this is not necessarily the case for the second system. Also for the second system we can shew that the three J circles are real; they are, in fact, the circles passing through one of the points A, B, C, and the antipoints of the remaining two.

All the preceding results may be extended to curves of the sixth order lying on a sphere by the simple process of inverting from a point outside the plane, and if we invert the equation (5), we see that the spherical curve lies on a cubic cone whose equation is of the form

$$\sqrt[3]{(l\alpha)} + \sqrt[3]{(m\beta)} + \sqrt[3]{(n\gamma)} = 0,$$

where α, β, γ are the perpendiculars from a point on three tangent planes to the sphere.

We may notice in conclusion a few degenerate forms of the plane curve. First, there are evidently three curves of the system which reduce to the fifth degree, as when we clear (5) of radicals and equate the coefficient of $(x^2+y^2)^3$ to zero, we get a cubic to determine the position of P on the circumscribing circle. These curves have each a real asymptote, and the points I, J for nodes; and it is easy to see that the general curve will be inverted from any point on itself into such a curve. If the curve be inverted from a focus, it will be transformed into a curve of the same nature of which one focus has gone off to infinity. The latter curves have the points I, J for triple points at which all the tangents coincide. If we take rectangular axes so that the coordinates of the foci are c, 0, $-c$, 0, the equation of the system (5) may be written in this case

$$(x^2 + y^2 - 2\alpha x - 3c^2)^3 + 27\,(c^2 - \alpha^2)\,\rho^2\rho'^2 = 0,$$

and that of the orthogonal system

$$(x^2 + y^2 - 2\beta y + 3c^2)^3 + 27\,(\beta^2 + c^2)\,\rho^2\rho'^2 = 0,$$

where $$\rho^2\rho'^2 = (x^2 + y^2)^2 - 2c^2\,(x^2 - y^2) + c^4.$$

If the general curve (5) be inverted from a node, it will be transformed into the curve whose equation is

$$BC\rho_1^{\frac{2}{3}} + CA\rho_2^{\frac{2}{3}} + AB\rho_3^{\frac{2}{3}} = 0,$$

where A, B, C lie on a line. This latter curve, it is easy to see, is a binodal quartic passing through I, J.

EXPRESSIONS FOR THE FIRST FIVE POWERS OF THE SERIES IN WHICH THE COEFFICIENTS ARE THE SUMS OF THE DIVISORS OF THE EXPONENTS.

By *J. W. L. Glaisher.*

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§1. In vol. xv. pp. 156–163* it was shown that

$$\frac{4KE+4KG+4KI}{\pi^2}=1-24\Sigma_1^\infty \sigma(n)\,q^{2n},$$

and $$\left(\frac{4KE+4KG+4KI}{\pi^2}\right)^2=1+48\Sigma_1^\infty \{5\sigma_3(n)-6n\sigma(n)\}\,q^{2n};$$

and, by means of these results, the square of the series $\Sigma_1^\infty \sigma(n)\,x^n$ was found to be equal to the series

$$\tfrac{1}{12}\Sigma_1^\infty \{5\sigma_3(n)-6n\sigma(n)+\sigma(n)\}x^n.$$

I have since obtained the q-series for the third, fourth and fifth powers of the quantity

$$\frac{4KE+4KG+4KI}{\pi^2},$$

from which the expressions for the corresponding powers of the series $\Sigma_1^\infty \sigma(n)\,x^n$ are at once derivable. These results are given in the following sections.

§2. Let P denote

$$\frac{4KE+4KG+4KI}{\pi^2},$$

and let $\sigma_s(n)$ denote the sum of the s^{th} powers of the divisors of n; then

$$P=1-24\Sigma_1^\infty \sigma(n)\,q^{2n},$$
$$P^2=1+48\Sigma_1^\infty \{5\sigma_3(n)-6n\sigma(n)\}\,q^{2n},$$
$$P^3=1-72\Sigma_1^\infty \{7\sigma_5(n)-30n\sigma_3(n)+24n^2\sigma(n)\}\,q^{2n},$$
$$P^4=1+96\Sigma_1^\infty \{5\sigma_7(n)-42n\sigma_5(n)+108n^2\sigma_3(n)-72n^3\sigma(n)\}\,q^{2n},$$
$$P^5=1-24\Sigma_1^\infty \{11\sigma_9(n)-150n\sigma_7(n)+720n^2\sigma_5(n)-1440n^3\sigma_3(n)$$
$$+864n^4\sigma(n)\}\,q^{2n}.$$

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* The following errata occur in this paper:

p. 156, last line, *for* $(5n^3+1)\,x^n$ *read* $(5n^3+n)\,x^n$,

p. 158, §6, *for* $\{12\Sigma_1^\infty \sigma(n)\,x^n\}^2$ *read* $12\,\{\Sigma_1^\infty \sigma(n)\,x^n\}^2$,

p. 163, note, *for* 28^{12} *read* $28x^{12}$.

Putting, for brevity, $\Sigma\sigma_s$ in place of $\Sigma_1^\infty \sigma_s(n)\, q^{2n}$, we may conveniently express these equations in the form

$$P = 1 - 24\Sigma\sigma,$$
$$P^2 = 1 + 240\Sigma\sigma_3 - 288\Sigma n\sigma,$$
$$P^3 = 1 - 504\Sigma\sigma_5 + 2160\Sigma n\sigma_3 - 1728\Sigma n^2\sigma,$$
$$P^4 = 1 + 480\Sigma\sigma_7 - 4032\Sigma n\sigma_5 + 10368\Sigma n^2\sigma_3 - 6912\Sigma n^3\sigma,$$
$$P^5 = 1 - 264\Sigma\sigma_9 + 3600\Sigma n\sigma_7 - 17280\Sigma n^2\sigma_5 + 34560\Sigma n^3\sigma - 20736\Sigma n^4\sigma.$$

§3. From these equations it can be deduced that, if S denote the series

$$\sigma(1)\, x + \sigma(2)\, x^2 + \sigma(3)\, x^3 + \sigma(4)\, x^4 + \&c.,$$

then

$$S^2 = \tfrac{1}{12}\, \Sigma_1^\infty \{5\sigma_3(n) - (6n-1)\,\sigma(n)\}\, x^n,$$
$$S^3 = \tfrac{1}{192}\, \Sigma_1^\infty \{7\sigma_5(n) - 10\,(3n-1)\,\sigma_3(n) + (24n^2 - 12n + 1)\,\sigma(n)\}\, x^n,$$
$$S^4 = \tfrac{1}{3456}\, \Sigma_1^\infty \{5\sigma_7(n) - 21\,(2n-1)\,\sigma_5(n) + 3\,(36n^2 - 30n + 5)\,\sigma_3(n) - (72n^3 - 72n^2 + 18n - 1)\,\sigma(n)\}\, x^n,$$
$$S^5 = \tfrac{1}{331776}\, \Sigma_1^\infty \{11\sigma_9(n) - 50\,(3n-2)\,\sigma_7(n) + 30\,(24n^2 - 28n + 7)\,\sigma_5(n) - 20\,(72n^3 - 108n^2 + 45n - 5)\,\sigma_3(n) + (864n^4 - 1440n^3 + 720n^2 - 120n + 5)\,\sigma(n)\}\, x^n.$$

Denoting $\Sigma_1^\infty \sigma_s(n)\, x^n$ by $\Sigma\sigma_s$, these formulæ may be written:

$$12S^2 = 5\Sigma\sigma_3 - 6\Sigma n\sigma + \Sigma\sigma,$$
$$192S^3 = 7\Sigma\sigma_5 - 30\Sigma n\sigma_3 + 24\Sigma n^2\sigma + 10\Sigma\sigma_3 - 12\Sigma n\sigma + \Sigma\sigma,$$
$$3456S^4 = 5\Sigma\sigma_7 - 42\Sigma n\sigma_2 + 108\Sigma n^2\sigma_3 - 72\Sigma n^3\sigma + 21\Sigma\sigma_5 - 90\Sigma n\sigma_3 + 72\Sigma n^2\sigma + 15\Sigma\sigma_3 - 18\Sigma n\sigma + \Sigma\sigma,$$

$$\begin{aligned}331776S^5 = 11\Sigma\sigma_9 &- 150\Sigma n\sigma_7 + 720\Sigma n^2\sigma_5 - 1440\Sigma n^3\sigma_3 + 864\Sigma n^4\sigma\\ &+100\Sigma\ \sigma_7 - 840\Sigma\ n\sigma_5 + 2160\Sigma n^2\sigma_3 - 1440\Sigma n^3\sigma\\ &+210\Sigma\ \sigma_5 - 900\Sigma\ n\sigma_3 + 720\Sigma n^2\sigma\\ &+100\Sigma\ \sigma_3 - 120\Sigma\ n\sigma\\ &+5\Sigma\ \sigma.\end{aligned}$$

The algebraical forms of these equations may be readily obtained, since

$$S = \Sigma\sigma \quad = \Sigma_1^\infty \frac{nx^n}{1-x^n},$$

$$\Sigma\sigma_s \ = \Sigma_1^\infty \frac{n^s x^n}{1-x^n},$$

$$\Sigma n\sigma_s = x\frac{d}{dx}\ \Sigma\ \sigma_s = \Sigma_1^\infty \frac{n^{s+1}x^n}{(1-x^n)^2},$$

$$\Sigma n^2\sigma_s = x\frac{d}{dx}\ \Sigma n\sigma_s = \Sigma_1^\infty \frac{n^{s+2}x^n(1+x^n)}{(1-x^n)^3},$$

&c. &c.

§4. In connexion with the paper 'On certain sums of products of quantities depending upon the divisors of a number' (pp. 1–20 of the present volume), I may mention that I have found that, denoting $n-r$ by r',

$$12\Sigma_{r=1}^{r=n} rr'\sigma(r)\,\sigma(r') = n^2\sigma_3(n) - n^3\sigma(n).$$

For example, putting $n = 6$,

$$\begin{aligned}12\{1.5.\sigma(1)\,\sigma(5) + 2.4.\sigma(2)\,\sigma(4) + 3.3.\sigma(3)\,\sigma(3)&\\ + 4.2.\sigma(4)\,\sigma(2) + 5.1.\sigma(5)\,\sigma(1)\}&\\ = 6^2\sigma_3(6) - 6^3\sigma(6),&\end{aligned}$$

which is readily verified.

I have also found that

$$120\Sigma_{r=1}^{r=n}\sigma_3(r)\,\sigma_3(r') = \sigma_7(n) - \sigma_3(n),$$

$$5040\Sigma_{r=1}^{r=n}\sigma_3(r)\,\sigma_5(r') = 11\sigma_9(n) - 21\sigma_5(n) + 10\sigma_3(n),$$

$$10080\Sigma_{r=1}^{r=n}\sigma_5(n)\,\sigma_7(n) = \sigma_{13}(n) + 20\sigma_7(n) - 21\sigma_5(n),$$

$$2640\Sigma_{r=1}^{r=n}\sigma_3(n)\,\sigma_9(n) = \sigma_{13}(n) - 11\sigma_9(n) + 10\sigma_3(n);$$

or, in the notation of the paper referred to,

$$120\Sigma\sigma_3\sigma_3 = \sigma_7(n) - \sigma_3(n),$$
$$5040\Sigma\sigma_3\sigma_5 = 11\sigma_9(n) - 21\sigma_5(n) + 10\sigma_3(n),$$
$$10080\Sigma\sigma_5\sigma_7 = \sigma_{13}(n) + 20\sigma_7(n) - 21\sigma_5(n),$$
$$2640\Sigma\sigma_3\sigma_9 = \sigma_{13}(n) - 11\sigma_9(n) + 10\sigma_3(n).$$

§ 5. It is perhaps worthy of remark that if $\Sigma fF = A$, then

$$\Sigma_{r=1}^{r=n} rf(r)\,F(r') = \Sigma_{r=1}^{r=n} r'f(r)\,F(r') = \tfrac{1}{2}nA.$$

This is at once seen to be true by combining the first and last terms, the second and the last but one, and so on.

Thus, for example, from the formula

$$12\Sigma\sigma\sigma = 5\sigma_3(n) - 6n\sigma(n) + \sigma(n),$$

we deduce

$$12\Sigma r\sigma(r)\,\sigma(r') = 5n\sigma_3(n) - 6n^2\sigma(n) + n\sigma(n),$$

and from the formula for $\Sigma rr'\sigma\sigma$ in the last section we deduce

$$24\Sigma_{r=1}^{r=n} r^2r'\sigma(r)\,\sigma(r') = n^3\sigma_3(n) - n^4\sigma(n).$$

ON THE SO-CALLED D'ALEMBERT-CARNOT GEOMETRICAL PARADOX.

By *Christine Ladd-Franklin.*

If I may be permitted to enter a discussion between Professor Cayley and Professor Sylvester (p. 113 and p. 92), I would suggest that there is a simple consideration which has not occurred to either of them, by means of which the obscurity in the problem in question is readily cleared up.

The problem is: from a point K, outside a given circle, to draw a line KMN such that the intercept, MN, made on it by the circle shall be equal to a given line, c.

If the line through K and the centre of the circle meets the circle in the points A and B, and if $KA = a$, $KB = b$, then, putting x for KM, we have

$$ab = x(x + c),$$
$$x = -\tfrac{1}{2}c \pm \tfrac{1}{2}\sqrt{(c^2 + 4ab)};$$

and hence $\quad x + C = \tfrac{1}{2}c \pm \tfrac{1}{2}\sqrt{(c^2 + 4ab)}.$

But, since the quantities a and b enter the equation only as their product, it is impossible to say whether they were both positive or both negative; that is to say, the statement of condition, (1), fails to discriminate between a circle to the right of K and a circle to the left of K. This is the ambiguity which has been introduced into the question, and this is exactly the ambiguity which the double sign in the solution is fitted to cover. If we take the upper sign in both cases, we have

$$x = \tfrac{1}{2}\sqrt{(c^2 + 4ab)} - \tfrac{1}{2}c = KM,$$

$$x + c = \tfrac{1}{2}\sqrt{(c^2 + 4ab)} + \tfrac{1}{2}c = KN.$$

If we take the lower sign, we have the solution for the circle on the other side of K (see fig. 1),

$$x = -\tfrac{1}{2}\sqrt{(c^2 + 4ab)} - \tfrac{1}{2}c = KM,'$$

$$x + c = -\tfrac{1}{2}\sqrt{(c^2 + 4ab)} + \tfrac{1}{2}c = KN.'$$

In both cases, the absolute values and the signs are the same whether the line is taken above AB or below AB, on KMN or KPQ.

This view of the question is confirmed by observing what happens when K is within the circle. In this case, a and b have opposite signs, and the equation becomes

$$-ab = x(x - c);$$

whence

$$x = \tfrac{1}{2}c \pm \tfrac{1}{2}\sqrt{(c^2 - 4ab)},$$

$$x - c = -\tfrac{1}{2}c \pm \tfrac{1}{2}\sqrt{(c^2 - 4ab)}.$$

Only one circle is possible, but x has two different absolute values according as it falls on one side or the other of the perpendicular to AB through K. We have

$$x = \tfrac{1}{2}c + \tfrac{1}{2}\sqrt{(c^2 - 4ab)} = KM,$$

$$x - c = -\tfrac{1}{2}c + \tfrac{1}{2}\sqrt{(c^2 - 4ab)} = KN,$$

and

$$x = \tfrac{1}{2}c - \tfrac{1}{2}\sqrt{(c^2 - 4ab)} = KM,'$$

$$x - c = -\tfrac{1}{2}c - \tfrac{1}{2}\sqrt{(c^2 - 4ab)} = KN.'$$

It should be observed that if in fig. 1 KM is x and KN is $x + c$, then if of the two lines KM' and KN' one is to be x and the other $x + c$, KM' must be x and KN' must be $x + c$.

ON SMALL MOTIONS OF SYSTEMS WITH A SINGLE DEGREE OF FREEDOM.

By *E. B. Elliott, M.A.*

Our attention is confined to a system for which there is an equation of energy

$$2T \equiv f(x)\,\dot{x}^2 = 2\phi(x) + c \text{..............} (1),$$

which does not involve the time explicitly.

The familiar theory of such systems tells us that the condition for the position $x = a$ to be one of equilibrium is $\phi'(a) = 0$, that the equilibrium in that position is stable or unstable according as $\phi''(a)$ and $f(a)$ have opposite or like signs, and that in the former case the time of a small oscillation about the position in question is $2\pi\sqrt{\left\{\dfrac{-f(a)}{\phi''(a)}\right\}}$.

Cases in which $\phi''(a) = 0$ are intermediate to those above instanced of obvious stability and instability, *i.e.* they are to the first approximation neutral. It is to such cases, and in particular to those of them in which there is real stability, that notice is now directed.

Differentiation of (1) gives, upon division by $\dot{x}$,

$$2f(x)\,\ddot{x} + f'(x)\,\dot{x}^2 = 2\phi'(x) \text{..............} (2),$$

which, upon consideration of a first special motion of the system, viz. rest in a position of equilibrium, gives the first criterion

$$0 = \phi'(a)$$

for the determination of such positions.

To discuss a small motion about such a position $x = a$, put $x = a + \zeta$, and expand, thus obtaining from (2)

$$2\{f(a) + \zeta f'(a) + \ldots\}\,\ddot{\zeta} + \{f'(a) + \zeta f''(a) + \ldots\}\,\dot{\zeta}^2$$
$$= 2\left\{\zeta\phi''(a) + \frac{\zeta^2}{2\,!}\phi'''(a) + \ldots\right\} \text{......} (3).$$

Now, ζ being in the motion considered very small, the only important term in each bracket will be the lowest one whose coefficient does not vanish. Moreover $\dot{\zeta}^2$ is always of a higher order in small quantities than $\ddot{\zeta}$, and so may in com-

parison be neglected, for $\ddot{\zeta} = \frac{1}{2} \frac{d(\dot{\zeta}^2)}{d\zeta}$, and so is of one lower dimentions in ζ, the standard small quantity involved, then is $\dot{\zeta}^2$ For small motions, therefore, this equation (3) reduces to

$$f(a)\,\ddot{\zeta} = \phi^{(n)}(a)\,\frac{\zeta^{n-1}}{n-1\,!} \;\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where $\phi^{(n)}(a)$ is the result of replacing x by a in the earliest derived function of x which does not vanish for this value of x.

Three cases now arise, viz.:

(α) The integer n may be odd $(=2p+1)$; in this case the equation (4) of the motion has the form $\ddot{\zeta} = \mu\zeta^{2p}$, so that $\ddot{\zeta}$ is always of the same sign for both positive and negative values of ζ. A displacement, therefore, in one of the two opposite directions from the central position $\zeta = 0$ will be followed by a restitution, but one in the other direction will be continued to a distance outside the range of our approximation. Consequently in this case the position of equilibrium is one of real instability. The time of return from a given displacement towards that side of the central position on which ζ has the reverse sign to that of $\ddot{\zeta}$ or $\phi^{(2p+1)}(a) \div f(a)$ will, however, be presently obtained.

(β) n may be even $(=2p)$, and $\phi^{(n)}(a) \div f(a)$ be positive. This will make $\ddot{\zeta}$ of like sign with ζ for displacements on both sides of $\zeta = 0$, and the position of equilibrium will be one of complete instability.

(γ) n may be even $(=2p)$, and $\phi^{(n)}(a) \div f(a)$ be negative. In this case $\ddot{\zeta}$ will always in the range of positions considered be of unlike sign to ζ; the tendency after any small displacement will be towards restitution, and, as a consequence, the equilibrium completely stable.

Taking, then, this case of complete stability, the equation (4) of a small motion becomes

$$\ddot{\zeta} = \frac{\phi^{(2p)}(a)}{f(a)}\,\frac{\zeta^{2p-1}}{2p-1\,!} = -\,c^2\zeta^{2p-1}, \text{ say.}$$

Suppose h the value of ζ (or $x-a$) corresponding to the extreme displacement, then, by integration,

$$\dot{\zeta}^2 = \frac{c^2}{p}\,(h^{2p} - \zeta^{2p}):$$

whence the time of descent to the central position $\zeta=0$, *i.e.* the time of a quarter oscillation, is

$$\tau=\frac{\sqrt{(p)}}{c}\int_0^h\frac{d\zeta}{\sqrt{(h^{2p}-\zeta^{2p})}}$$

$$=\frac{\sqrt{(p)}}{ch^{p-1}}\,\frac{\Gamma\left(\frac{1}{2p}\right)\Gamma(\frac{1}{2})}{2p\Gamma\left(\frac{1}{2p}+\frac{1}{2}\right)}$$

$$=\frac{1}{h^{p-1}}\,\frac{\Gamma\left(\frac{1}{2p}\right)}{\Gamma\left(\frac{1}{2p}+\frac{1}{2}\right)}\left\{-\frac{\Gamma(2p)f(a)\pi}{4p\phi^{(2p)}(a)}\right\}^{\frac{1}{2}}\ldots\ldots.(A),$$

and consequently varies inversely as the $(p-1)^{\text{th}}$ power of the extent of the displacement. In particular if the disturbance be an indefinitely small one, the time of restitution is infinite.

In exactly the same way, returning to the case of (α) above, the time of return to the position of neutral but unstable equilibrium from a displacement h in that sense which makes $\ddot{\zeta}$ of the opposite sign to h is

$$\left(\frac{1}{h^2}\right)^{\frac{1}{2}p}\frac{\Gamma\left(\frac{1}{2p+1}\right)}{\Gamma\left(\frac{1}{2p+1}+\frac{1}{2}\right)}\left\{-\frac{\Gamma(2p+1)f(a)\pi h}{2(2p+1)\phi^{(2p+1)}(a)}\right\}^{\frac{1}{2}},$$

the expression under the second root being certainly positive, since h and $\phi^{(2p+1)}(a)\div f(a)$ have opposite signs.

Queen's College, Oxford,
March, 1885.

NOTE ON THE WAVE SURFACE.

By Prof. *A. Mannheim.*

Un ellipsoïde a pour centre le point o (fig. 3), m est un point de cette surface, et mn la normale en ce point.

Prenons pour plan de la figure le plan omn. Dans ce plan élevons du point o à om la perpendiculaire om_1 sur laquelle nous portons le segment om_1 égal à om.

Lorsque le point m se déplace sur l'ellipsoïde le point m_1 décrit une surface de l'onde, et l'on sait que la normale en m_1

à cette surface est la perpendiculaire abaissée de ce point sur mn; c'est-à-dire qu'en faisant tourner le plan de la figure sur lui-même autour du point o de façon que le point m vienne en m_1, la normale mn vient prendre la position de la normale à la surface de l'onde.

Lorsque m décrit une courbe, le point m_1 décrit une courbe correspondante. Je suppose que le point m décrive une ligne de courbure (m) de l'ellipsoïde, et je vais chercher une propriété géométrique de la ligne correspondante décrite (m_1).

Pour cela, il suffit de transformer une propriété géométrique de (m); c'est ainsi que j'ai opéré en 1877 au congrès du Havre, lorsque j'ai transformé cette propriété :

Si du centre d'un ellipsoïde on mène des plans parallèles aux plans tangents à cette surface aux différents points d'une ligne de courbure, ces plans enveloppent un cône du 2^d degré qui coupe l'ellipsoïde suivant une courbe sphérique.

On arrive ainsi à ce théorême :

Les diamètres de la surface de l'onde abaissés perpendiculairement sur les normales à cette surface, dont les pieds sont des points de (m_1), sont égaux entre eux.

Maintenant transformons la propriété suivante :

Les normales à l'ellipsoïde, dont les pieds sont des points d'une ligne de courbure, rencontrent l'un ou l'autre des plans principaux suivant une conique.

Soit rr_1 la trace d'un plan principal de l'ellipsoïde sur le plan omn. Les points tels que r appartiennent à une conique de centre o.

Lorsque le plan de la figure tourne sur lui-même d'un angle droit autour de o, le point r vient en r' sur la normale en m_1 à la surface de l'onde. On a $or' = or \operatorname{tang} m_1 r_1 o$. On peut donc dire que l'on obtient le point r sur le diametre de la surface de l'onde qui passe par la trace r_1 de la normale en m_1, en portant un segment égal à $or \operatorname{tang} m_1 r_1 o$. On a alors ce théorême :

On mène les normales à la surface de l'onde, dont les pieds sont des points d'une courbe (m_1). On joint la trace d'une de ces normales sur l'un des plans principaux au centre de la surface, et l'on porte sur cette droite (de part et d'autre du centre) un segment égal au segment compris entre le centre et cette trace, multiplié par la tangente de l'angle que la normale fait avec ce segment. Le lieu des extrémités des segments ainsi portés est une conique.

ON THE RATIONALIZATION OF $a^{\frac{1}{n}}+b^{\frac{1}{n}}+c^{\frac{1}{n}}=0.$

By *Piers C. Ward.*

1. To rationalize $a^{\frac13}+b^{\frac13}+c^{\frac13}=0$

Assume $\quad x^3=a \quad y^3=b \quad z^3=c,$

also $\quad y=mx \quad z=nx,$

then $\quad 1+m+n=0,$

$$am^3=b,$$

$$an^3=c.$$

Hence the equations

$$\begin{aligned} a+b+c+ \quad 3am \quad &+ \quad 3am^2 &=0,\\ 3b \quad + (a+b+c)\,m &+ \quad 3am^2 &=0,\\ 3b \quad + \quad 3bm \quad &+(a+b+c)\,m &=0; \end{aligned}$$

therefore
$$\begin{vmatrix} a+b+c, & 3a \;, & 3a \\ 3b \;, & a+b+c, & 3a \\ 3b \;, & 3b \;, & a+b+c \end{vmatrix}=0.$$

If we let $m+1=\lambda$ and eliminate we see that

$$\begin{vmatrix} a+b+c, & -3a \;, & 3a \\ -3c \;, & a+b+c, & -3a \\ 3c \;, & 3c \;, & a+b+c \end{vmatrix} \equiv (a+b+c)^3-27abc$$

$$\equiv \left\{a+b+c+3a\left(\frac{b}{a}\right)^{\frac13}+3a\left(\frac{b}{a}\right)^{\frac23}\right\}\left\{a+b+c+3a\omega\left(\frac{b}{a}\right)^{\frac13}\omega^2+3a\left(\frac{b}{a}\right)^{\frac23}\right\}$$

$$\left\{a+b+c+3a\omega^2\left(\frac{b}{a}\right)^{\frac13}+3a\omega\left(\frac{b}{a}\right)^{\frac23}\right\} \text{ where } \omega^3=1;$$

this product may obviously be expressed in eleven other forms.
If we assume $a+b+c=3x$ we obtain the identity

$$x^3-3abx+ab\,(a+b)\equiv\left\{x+a\left(\frac{b}{a}\right)^{\frac13}+a\left(\frac{b}{a}\right)^{\frac23}\right\}\times$$

$$\left\{x+a\omega\left(\frac{b}{a}\right)^{\frac13}+a\omega^2\left(\frac{b}{a}\right)^{\frac23}\right\}\left\{x+a\omega^2\left(\frac{b}{a}\right)^{\frac13}+a\omega\left(\frac{b}{a}\right)^{\frac23}\right\}.$$

Since every cubic may be reduced to the form $x^3 + qx + r = 0$ and hence to the form $x^3 - 3abx + ab(a+b) = 0$ we have a solution of the general cubic.

2. Making the same assumptions as before we find if

$$a^{\frac{1}{5}} + b^{\frac{1}{5}} + c^{\frac{1}{5}} = 0;$$

then

$$\begin{vmatrix} a+b+c, & 5a, & 10a, & 10a, & 5a \\ 5b, & a+b+c, & 5a, & 10a, & 10a \\ 10b, & 5b, & a+b+c, & 5a, & 10a \\ 10b, & 10b, & 5b, & a+b+c, & 5a \\ 5b, & 10b, & 10b, & 5b, & a+b+c \end{vmatrix}$$

$$\equiv (a+b+c)^5 - (5)^4 abc(a+b+c)^2 + (5)^5 abc(bc+ca+ab)$$

$\equiv \Pi = 0$ where $\Pi \equiv f(\alpha_1) \times f(\alpha_2) \ldots \times f(\alpha_5)$,

$$\text{and } f(m) \equiv a(m+1)^5 + c,$$

$$\alpha_1, \alpha_2, \ldots \alpha_5 \text{ being roots of } am^5 = b.$$

If we assume $a+b+c = 5x$, we see that

$$x^5 - 25abx^3 + 30ab(a+b)x^2 - 5abx\{2(a+b)^2 - ab\} + ab(a+b)\{(a+b)^2 - ab\}$$

contains

$$x + a\left(\frac{b}{a}\right)^{\frac{4}{5}} + 2a\left(\frac{b}{a}\right)^{\frac{3}{5}} + 2a\left(\frac{b}{a}\right)^{\frac{2}{5}} + a\left(\frac{b}{a}\right)^{\frac{1}{5}}$$

as a factor. Hence the following quintic may be solved,

$$x^5 - 25qx^3 + 30pqx^2 - 5q(2p^2 - q)x + pq(p^2 - q) = 0.$$

The rationalization of $a^{\frac{1}{n}} + b^{\frac{1}{n}} + c^{\frac{1}{n}} = 0$ is easily seen to be equivalent to the eliminant of

$$a(m+1)^n = (-1)^n c \text{ and } am^n = b.$$

Since the result must be symmetrical with respect to a, b and c, it follows that the eliminant may be expressed in twelve equivalent forms as a determinant of the n^{th} order, which may be resolved into n factors, and expressed in terms of $a+b+c$, $bc+ca+ab$, and abc.

3. To rationalize $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} + d^{\frac{1}{3}} = 0$,

let $$a^{\frac{1}{3}} + b^{\frac{1}{3}} + x^{\frac{1}{3}} = 0,$$

then $$c^{\frac{1}{3}} + d^{\frac{1}{3}} - x^{\frac{1}{3}} = 0;$$

therefore $$(x + a + b)^3 - 27abx = 0,$$

and $$(x - c - d)^3 - 27cdx = 0.$$

Hence eliminating x we obtain a rational relation connecting a, b, c and d. By Bezout's method we obtain the result in the form of a determinant of the third order, from this may be derived several peculiar identities.

A PARTICULAR METHOD FOR THE SOLUTION OF SOME LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

By *A. R. Forsyth.*

When in the linear differential equation

$$\frac{d^2v}{dx^2} + Iv = 0,$$

the quantity I is a rational algebraical function of a fractional form, the following method is sometimes effective for the solution of the equation.

Let a quantity $ze^{\int P_1 dx}$ be substituted for v; the equation then becomes

$$\frac{d^2z}{dx^2} + 2P_1\frac{dz}{dx} + P_2 z = 0,$$

where $$P_2 = I + P_1^2 + \frac{dP_1}{dx}.$$

On integrating the equation as if the left-hand side were a perfect differential, we have

$$\frac{dz}{dx} + 2P_1 z + \int z\left(P_2 - 2\frac{dP_1}{dx}\right) dx = A.$$

Since the quantities P_1 and P_2 are as yet connected by only a single relation, we may assign, as a further relation to determine them, the equation

$$P_2 = 2\frac{dP_1}{dx};$$

this, in combination with the former, gives as the equation to determine P_1

$$\frac{dP_1}{dx} - P_1^2 = I.$$

If any value of P_1 satisfying this be obtained, then a first integral of the original equation is obtained in the form

$$\frac{dz}{dx} + 2P_1 z = A.$$

With the assumption which was made as to the form of I we may write

$$I = \frac{V}{T^2U} = \frac{VU}{T^2U^2} = \frac{VU}{\phi^2},$$

where T, U, V are rational, integral and algebraical functions of x. Then we may assume

$$P_1 = \frac{f(x)}{\phi},$$

leaving the constants in $f(x)$ as the quantities to be determined after this value of P_1 is substituted in the equation; but in general there are not sufficient disposable constants in $f(x)$ to allow the equation to be satisfied. Hence this method, like other methods proposed for the solution of the linear equation of the second order, is not one of universal application, but proves effective only in particular cases. And it should be pointed out that the utility of the method depends on the form of the equation which gives P_1; this would be lost by the substitution

$$P_1 = -\frac{1}{w}\frac{dw}{dx},$$

for then the equation giving P_1 becomes changed to the original equation.

As an example, consider under what conditions the equation

$$\frac{1}{v}\frac{d^2v}{dx^2} = \frac{\mathrm{a}}{(x-a)^2} + \frac{\mathrm{b}}{(x-b)^2} + \frac{\mathrm{c}}{(x-c)^2} + \frac{\mathrm{a}-\mathrm{b}-\mathrm{c}}{(x-b)(x-c)}$$
$$+ \frac{\mathrm{b}-\mathrm{c}-\mathrm{a}}{(x-c)(x-a)} + \frac{\mathrm{c}-\mathrm{a}-\mathrm{b}}{(x-a)(x-b)}$$

can be integrated by the foregoing method. The expression on the right-hand side of this equation is the value of $-I$; and we have to determine P_1 so as to satisfy

$$\frac{dP_1}{dx} - P_1^2 = I.$$

Let
$$P_1 = \frac{e}{x-a} + \frac{f}{x-b} + \frac{g}{x-c};$$

then
$$P_1^2 - \frac{dP_1}{dx} = \frac{e^2+e}{(x-a)^2} + \ldots + \frac{2ef}{(x-a)(x-b)} + \ldots,$$

and the assumed value for P_1 will be suitable if the constants e, f, g be such as to satisfy the relations

$$e^2 + e = \mathrm{a},$$
$$f^2 + f = \mathrm{b},$$
$$g^2 + g = \mathrm{c},$$
$$2ef = \mathrm{c} - \mathrm{a} - \mathrm{b},$$
$$2fg = \mathrm{a} - \mathrm{b} - \mathrm{c},$$
$$2ge = \mathrm{b} - \mathrm{c} - \mathrm{a}.$$

From the first three of these we have

$$e + \tfrac{1}{2} = (\mathrm{a} + \tfrac{1}{4})^{\frac{1}{2}},$$
$$f + \tfrac{1}{2} = (\mathrm{b} + \tfrac{1}{4})^{\frac{1}{2}},$$
$$g + \tfrac{1}{2} = (\mathrm{c} + \tfrac{1}{4})^{\frac{1}{2}};$$

and the relation obtained by the addition of the six equations gives, either

$$e + f + g = 0,$$

or
$$e + f + g = -1.$$

The latter is easily proved to be the necessary relation; it is equivalent to

$$(\mathrm{a}+\tfrac{1}{4})^{\frac{1}{2}}+(\mathrm{b}+\tfrac{1}{4})^{\frac{1}{2}}+(\mathrm{c}+\tfrac{1}{4})^{\frac{1}{2}}=\tfrac{1}{2},$$

which must be satisfied by some one set of signs given to the radicals.

When this condition is satisfied, the value of P_1 is given by

$$P_1=\frac{(\mathrm{a}+\frac{1}{4})^{\frac{1}{2}}-\frac{1}{2}}{x-a}+\frac{(\mathrm{b}+\frac{1}{4})^{\frac{1}{2}}-\frac{1}{2}}{x-b}+\frac{(\mathrm{c}+\frac{1}{4})^{\frac{1}{2}}-\frac{1}{2}}{x-c};$$

and it is not difficult to prove that the primitive of the original equation in v is

$$v=Ae^{\int P_1 dx}+Be^{-\int P_1 dx},$$

where A and B are arbitrary constants.

The differential equation of the hypergeometric series is reduced from

$$\frac{d^2y}{dx^2}+\frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx}-\frac{\alpha\beta}{x(1-x)}y=0$$

to its normal form

$$\frac{d^2v}{dx^2}+Iv=0,$$

by the substitution

$$v=yx^{\frac{1}{2}\gamma}(1-x)^{\frac{1}{2}(\alpha+\beta+1-\gamma)}.$$

The quantity I is

$$\frac{\frac{1}{4}(1-\lambda^2)}{x^2}+\frac{\frac{1}{4}(1-\nu^2)}{(x-1)^2}+\frac{\frac{1}{4}(\lambda^2-\mu^2+\nu^2-1)}{x(x-1)}$$

where

$$\lambda^2=(1-\gamma)^2,\quad \mu^2=(\alpha-\beta)^2,\quad \nu^2=(\gamma-\alpha-\beta)^2;$$

and this is therefore a particular case of the example already considered, being obtained by making $a=0$, $b=1$, $c=\infty$, and

$$\mathrm{a}=\tfrac{1}{4}(\lambda^2-1),\qquad \mathrm{b}=\tfrac{1}{4}(\nu^2-1),\qquad \mathrm{c}=\tfrac{1}{4}(\mu^2-1).*$$

The condition that the equation can be integrated by the above method is therefore

$$\lambda+\mu+\nu=1,$$

* Cayley, *Camb. Phil. Trans.* (1881), vol. XIII.

for any set of signs derived from the radicals. When we consider all the possible cases of this condition by substituting the values of λ, μ, ν in terms of α, β, γ, we find that they reduce themselves to four independent cases, viz.,

$$\beta = 0,$$

$$\beta = 1,$$

$$\beta = \gamma,$$

$$\beta = \gamma - 1.$$

The first of these must be rejected since the differential equation loses the term in y; in the remaining three cases the differential equations are

$$\text{(i)} \quad \frac{d^2y}{dx^2} + \frac{\gamma - (\alpha + 2)\,x}{x\,(1-x)}\,\frac{dy}{dx} - \frac{\alpha}{x\,(1-x)}\,y = 0,$$

$$\text{(ii)} \quad \frac{d^2y}{dx^2} + \frac{\gamma - (\alpha + \gamma + 1)\,x}{x\,(1-x)}\,\frac{dy}{dx} - \frac{\alpha\gamma}{x\,(1-x)}\,y = 0,$$

$$\text{(iii)} \quad \frac{d^2y}{dx^2} + \frac{\gamma - (\alpha + \gamma)\,x}{x\,(1-x)}\,\frac{dy}{dx} - \frac{\alpha\,(\gamma - 1)}{x\,(1-x)}\,y = 0.$$

The corresponding integrals are respectively

$$\text{(i)} \quad x\,(1-x)\,y = A + Bx^{2-\gamma}\,(1-x)^{\gamma-\alpha},$$

$$\text{(ii)} \quad x\,(1-x)^{\alpha}y = A + Bx^{2\ \gamma}\,(1-x)^{\alpha},$$

$$\text{(iii)} \quad x^{\gamma}\,(1-x)\,y = A + Bx^{\gamma}\,(1-x)^{1-\alpha}.$$

It may be remarked that, in the fifteen cases in which Schwarz (*Crelle*, t. LXXV) has obtained the integral of the hypergeometric series in a finite form, the quantities λ, μ, ν are such as to satisfy the inequality

$$\lambda + \mu + \nu > 1,$$

which is the necessary condition for the application of his method.

THE ADDITION THEOREM FOR THE SECOND AND THIRD ELLIPTIC INTEGRALS.

By *A. R. Forsyth.*

1. LET the elliptic functions of an argument u_r be denoted s_r, c_r, d_r and suppose that there are four arguments u connected by the relation

$$u_1 + u_2 + u_3 + u_4 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

then we know that an equation

$$As + Bc + Cd = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is satisfied for each of the four quantities u, the relation between the functions of which is given by

$$\begin{vmatrix} s_1, & c_1, & d_1, & 1 \\ s_2, & c_2, & d_2, & 1 \\ s_3, & c_3, & d_3, & 1 \\ s_4, & c_4, & d_4, & 1 \end{vmatrix} = 0.$$

Further, if $E(u)$ denote the second elliptic integral corresponding to the argument u, we have

$$E(u_1) + E(u_2) + E(u_3) + E(u_4) = -\frac{8k^2ABC}{(A^2+B^2+k^2C^2)^2 - 4k^2B^2C^2}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

and, if $\Pi(u, a)$ be the corresponding third elliptic integral after Jacobi, we have

$$\Pi(u_1, a) + \Pi(u_2, a) + \Pi(u_3, a) + \Pi(u_4, a)$$
$$= \tfrac{1}{2}\log\left[\frac{\{(A-k\sigma)^2 + (B\delta + Ck\gamma)^2\}\{(A+k\sigma)^2 + (B\delta - Ck\gamma)^2\}}{\{(A+k\sigma)^2 + (B\delta + Ck\gamma)^2\}\{(A-k\sigma)^2 + (B\delta - Ck\gamma)^2\}}\right]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where σ, γ, δ are the elliptic functions of the parametric argument a.*

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* A memoir on Abel's theorem and the Abelian functions, *Phil. Trans.* 1883, pp. 340, *et seq.*; a mistake in sign in equation (4) is here corrected.

2. For brevity, let

$$\begin{aligned}
\Delta\ &= (A^2 + B^2 + k^2C^2)^2 - 4k^2B^2C^2,\\
\Delta_1 &= (k^2 + A^2 + k'^2B^2)^2 - 4k^2A^2,\\
\Delta_2 &= (1 + A^2 - k'^2C^2)^2 - 4A^2,\\
\Delta_3 &= (1 - B^2 - C^2)^2 - 4B^2C^2.
\end{aligned}$$

In the various equations written below Σ indicates summation with regard to the four suffixes 1, 2, 3, 4; thus $\Sigma s_1s_2s_3$ denotes $s_1s_2s_3 + s_2s_3s_4 + s_3s_4s_1 + s_4s_1s_2$. When we rationalise equation (2) so that it becomes an equation of the fourth degree in s we find

$$\left.\begin{aligned}
\Delta\Sigma s_1 &= 4A\,(A^2 + B^2 + k^2C^2),\\
\Delta\Sigma s_1s_2s_3 &= 4A\,(1 - B^2 - C^2),\\
\Delta s_1s_2s_3s_4 &= \Delta_3.
\end{aligned}\right\}$$

The following results are similarly obtained

$$\left.\begin{aligned}
\Delta\Sigma c_1 &= 4B\,(A^2 + B^2 - k^2C^2),\\
\Delta\Sigma c_1c_2c_3 &= 4B\,(1 - A^2 - k'^2C^2,\\
\Delta c_1c_2c_3c_4 &= \Delta_2.
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\Delta\Sigma d_1 &= 4k^2C\,(A^2 - B^2 + k^2C^2),\\
\Delta\Sigma d_1d_2d_3 &= 4k^2C\,(k^2 - A^2 + k'^2B^2),\\
\Delta d_1d_2d_3d_4 &= \Delta_1.
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\Delta_3\Sigma\frac{c_1}{s_1} &= 4AB\,(1 - B^2 + C^2),\\
\Delta_2\Sigma\frac{s_1}{c_1} &= 4AB\,(1 - A^2 + k'^2C^2).
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\Delta_3\Sigma\frac{d_1}{s_1} &= 4AC\,(1 + B^2 - C^2),\\
\Delta_1\Sigma\frac{s_1}{d_1} &= 4AC\,(k^2 - A^2 - k'^2B^2).
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\Delta_2\Sigma\frac{d_1}{c_1} &= 4k'^2BC\,(1 + A^2 - k'^2C^2),\\
\Delta_1\Sigma\frac{c_1}{d_1} &= -\,4k'^2BC\,(k^2 + A^2 + k'^2B^2).
\end{aligned}\right\}$$

$$\left.\begin{aligned}\Delta\Delta_3\Sigma\frac{c_1d_1}{s_1} &= 8ABC(\Delta+k^2\Delta_3),\\ \Delta_1\Delta_2\Sigma\frac{s_1}{c_1d_1} &= 8ABC(\Delta_1+k^2\Delta_2).\end{aligned}\right\}$$

$$\left.\begin{aligned}\Delta\Delta_2\Sigma\frac{s_1d_1}{c_1} &= 8ABC(k'^2\Delta-k^2\Delta_2),\\ \Delta_1\Delta_3\Sigma\frac{c_1}{s_1d_1} &= 8ABC(\Delta_1-k^2k'^2\Delta_3).\end{aligned}\right\}$$

$$\left.\begin{aligned}\Delta\Delta_1\Sigma\frac{s_1c_1}{d_1} &= 8ABC(\Delta_1+k'^2\Delta),\\ \Delta_2\Delta_3\Sigma\frac{d_1}{s_1c_1} &= 8ABC(\Delta_2+k'^2\Delta_3).\end{aligned}\right\}$$

$$\Delta_1\Delta_2\Delta_3\Sigma\frac{1}{s_1c_1d_1} = 8ABC(\Delta_1\Delta_2+k^4\Delta_2\Delta_3+\Delta_3\Delta_1).$$

3. Consider now equation (3) which gives the sum of the four elliptic integrals; we may write it

$$\Delta\Sigma E(u_1) = -8k^2ABC.$$

The formulæ of § 2 then lead at once to the following seven expressions, equal to one another, as the value of $\Sigma E(u_1)$. They are

$$E(u_1)+E(u_2)+E(u_3)+E(u_4)$$

$$= -\frac{k^2s_1s_2s_3s_4}{1+k^2s_1s_2s_3s_4}\left(\frac{c_1d_1}{s_1}+\frac{c_2d_2}{s_2}+\frac{c_3d_3}{s_3}+\frac{c_4d_4}{s_4}\right)$$

$$= -\frac{k^2c_1c_2c_3c_4d_1d_2d_3d_4}{d_1d_2d_3d_4+k^2c_1c_2c_3c_4}\left(\frac{s_1}{c_1d_1}+\frac{s_2}{c_2d_2}+\frac{s_3}{c_3d_3}+\frac{s_4}{c_4d_4}\right)$$

$$= \frac{k^2c_1c_2c_3c_4}{k^2c_1c_2c_3c_4-k'^2}\left(\frac{s_1d_1}{c_1}+\frac{s_2d_2}{c_2}+\frac{s_3d_3}{c_3}+\frac{s_4d_4}{c_4}\right)$$

$$= \frac{k^2s_1s_2s_3s_4d_1d_2d_3d_4}{k^2k'^2s_1s_2s_3s_4-d_1d_2d_3d_4}\left(\frac{c_1}{s_1d_1}+\frac{c_2}{s_2d_2}+\frac{c_3}{s_3d_3}+\frac{c_4}{s_4d_4}\right)$$

$$= \frac{k^2d_1d_2d_3d_4}{k'^2+d_1d_2d_3d_4}\left(\frac{s_1c_1}{d_1}+\frac{s_2c_2}{d_2}+\frac{s_3c_3}{d_3}+\frac{s_4c_4}{d_4}\right)$$

$$= \frac{k^2 s_1 s_2 s_3 s_4 + c_1 c_2 c_3 c_4}{c_1 c_2 c_3 c_4 + k'^2 s_1 s_2 s_3 s_4} \left(\frac{d_1}{s_1 c_1} + \frac{d_2}{s_2 c_2} + \frac{d_3}{s_3 c_3} + \frac{d_4}{s_4 c_4} \right)$$

$$= -k^2 \frac{\dfrac{1}{s_1 c_1 d_1} + \dfrac{1}{s_2 c_2 d_2} + \dfrac{1}{s_3 c_3 d_3} + \dfrac{1}{s_4 c_4 d_4}}{\dfrac{1}{s_1 s_2 s_3 s_4} + \dfrac{1}{c_1 c_2 c_3 c_4} + \dfrac{k^4}{d_1 d_2 d_3 d_4}}.$$

By rationalising equation (2) so as to have in turn the quantities cs, sd, dc as the independent variables, we can obtain the following equations:

$$E(u_1) + E(u_2) + E(u_3) + E(u_4)$$

$$= \frac{k^2 \Sigma d_1 (c_1 - c_2 c_3 c_4) - \Sigma c_1 (d_1 - d_2 d_3 d_4)}{k'^2 \Sigma s_1}$$

$$= \frac{k^2 \Sigma c_1 (k^2 s_1 - s_2 s_3 s_4) + \Sigma s_1 (k'^2 c_1 - c_2 c_3 c_4)}{\Sigma d_1}$$

$$= \frac{k^2 \Sigma d_1 (s_1 - s_2 s_3 s_4) + \Sigma s_1 (k'^2 d_1 - d_2 d_3 d_4)}{\Sigma c_1}.$$

4. From the formulæ in § 2 we have

$$\Delta^2 \{\Sigma s_1 s_2 s_3\}^2 - 16 A^2 \Delta s_1 s_2 s_3 s_4 = 64 A^2 B^2 C^2,$$

and

$$\Delta^2 \{\Sigma s_1\}^2 \quad - 16 A^2 \Delta \quad = 64 k^2 A^2 B^2 C^2;$$

hence

$$\Delta^2 [\{\Sigma s_1 s_2 s_3\}^2 - s_1 s_2 s_3 s_4 \{\Sigma s_1\}^2] = 64 (1 - k^2 s_1 s_2 s_3 s_4) A^2 B^2 C^2.$$

It therefore follows that

$$\{\Sigma E(u_1)\}^2 = k^4 \frac{\{\Sigma s_1 s_2 s_3\}^2 - s_1 s_2 s_3 s_4 \{\Sigma s_1\}^2}{1 - k^2 s_1 s_2 s_3 s_4}.$$

The following formulæ may be similarly obtained:

$$\{\Sigma E(u_1)\}^2 = k^4 \frac{\{\Sigma c_1 c_2 c_3\}^2 - c_1 c_2 c_3 c_4 \{\Sigma c_1\}^2}{k'^2 + k^2 c_1 c_2 c_3 c_4}$$

$$= \frac{\{\Sigma d_1 d_2 d_3\}^2 - d_1 d_2 d_3 d_4 \{\Sigma d_1\}^2}{d_1 d_2 d_3 d_4 - k'^2}$$

$$= k^4 \frac{s_1 s_2 s_3 s_4 \{\Sigma s_1 c_2 c_3 c_4\}^2 - c_1 c_2 c_3 c_4 \{\Sigma c_1 s_2 s_3 s_4\}^2}{k'^2 s_1 s_2 s_3 s_4 - c_1 c_2 c_3 c_4}$$

$$= k^4 \frac{d_1 d_2 d_3 d_4 \{\Sigma d_1 s_2 s_3 s_4\}^2 - s_1 s_2 s_3 s_4 \{\Sigma s_1 d_2 d_3 d_4\}^2}{d_1 d_2 d_3 d_4 + k^2 k'^2 s_1 s_2 s_3 s_4}$$

$$= \frac{k^4}{k'^4} \frac{c_1 c_2 c_3 c_4 \{\Sigma c_1 d_2 d_3 d_4\}^2 - d_1 d_2 d_3 d_4 \{\Sigma d_1 c_2 c_3 c_4\}^2}{k^2 c_1 c_2 c_3 c_4 - d_1 d_2 d_3 d_4}.$$

5. By changing the arguments u by $\pm K$, it is possible to derive the expressions in § 3 from one another. From the ordinary addition-theorem for the second elliptic integral we have

$$E(u_1 \pm K) - E(u_1) \mp E = -k^2 \frac{s_1 c_1}{d_1},$$

where the upper signs and the lower must be taken together. If now two of the arguments u be changed by $+K$ and two by $-K$, so that the sum of the four is still zero, we have, on the addition of the four equations of the form just written down,

$$\Sigma E(u_1 + K) - \Sigma E(u_1) = -k^2 \Sigma \frac{s_1 c_1}{d_1}$$

$$= -\frac{k'^2 + d_1 d_2 d_3 d_4}{d_1 d_2 d_3 d_4} \Sigma E(u_1),$$

and therefore

$$\Sigma E(u_1 \pm K) = -\frac{k'^2}{d_1 d_2 d_3 d_4} \Sigma E(u_1).$$

When this equation is used it will be found that all the transformations of the expressions in § 3 become identities.

6. By means of the equations obtained it is possible to obtain the well-known formulæ connecting the functions of four arguments whose sum is zero. Thus it is at once verified that

$$k'^2 (\Delta - k^2 \Delta_3) + k^2 \Delta_2 - \Delta_1 = 0,$$

and this is equivalent to

$$k'^2 (1 - k^2 s_1 s_2 s_3 s_4) + k^2 c_1 c_2 c_3 c_4 - d_1 d_2 d_3 d_4 = 0,$$

a formula due to Gudermann.

Again, from the equation whose roots are s_1, s_2, s_3, s_4, we have

$$\Delta \Sigma s_1 s_2 = 4A^2 + 4(1+k^2) B^2 C^2 + 2(A^2 + B^2 + k^2 C^2)(1 - B^2 - C^2),$$

and by means of this it is easy to verify that

$$2 - \Sigma s_1^2 + k^2 \Sigma s_1^2 s_2^2 s_3^2 - 2k^4 s_1^2 s_2^2 s_3^2 s_4^2 = 2(c_1 c_2 c_3 c_4 - s_1 s_2 s_3 s_4 d_1 d_2 d_3 d_4),$$

which leads to

$$c_4 c_3 - c_1 c_2 + s_1 s_2 d_3 d_4 - d_1 d_2 s_3 s_4 = 0$$

and the two corresponding equations. The other results of the same type may be similarly obtained.

7. Some of the expressions for $\Sigma E(u_1)$ in § 3 can be derived from the ordinary addition-theorem for the second elliptic integral. Let

$$u_1 + u_2 = v = -(u_3 + u_4),$$

and let s denote $\operatorname{sn} v$. Then we have

$$E(u_1) + E(u_2) - E(v) = \quad k^2 s_1 s_2 s,$$
$$E(u_3) + E(u_4) + E(v) = -k^2 s_3 s_4 s,$$

and therefore

$$\Sigma E(u_1) = k^2 s\,(s_1 s_2 - s_3 s_4).$$

Now

$$s = \frac{s_1 c_2 d_2 + s_2 c_1 d_1}{1 - k^2 s_1{}^2 s_2{}^2} = \frac{-s_3 c_4 d_4 - s_4 c_3 d_3}{1 - k^2 s_3{}^2 s_4{}^2}$$

$$= \frac{s_1 s_2 s_3 s_4 \Sigma \dfrac{c_1 d_1}{s_1}}{(s_3 s_4 - s_1 s_2)(1 + k^2 s_1 s_2 s_3 s_4)},$$

or

$$s\,(s_1 s_2 - s_3 s_4) = -\frac{s_1 s_2 s_3 s_4}{1 + k^2 s_1 s_2 s_3 s_4} \Sigma \frac{c_1 d_1}{s_1}.$$

Again

$$s = \frac{s_1 c_1 d_2 + s_2 c_2 d_1}{c_1 c_2 + s_1 s_2 d_1 d_2} = \frac{-s_3 c_3 d_4 - s_4 c_4 d_3}{c_3 c_4 + s_3 s_4 d_3 d_4}$$

$$= \frac{d_1 d_2 d_3 d_4 \Sigma \dfrac{s_1 c_1}{d_1}}{c_1 c_2 d_3 d_4 - c_3 c_4 d_1 d_2 + d_1 d_2 d_3 d_4 (s_1 s_2 - s_3 s_4)}$$

$$= \frac{d_1 d_2 d_3 d_4 \Sigma \dfrac{s_1 c_1}{d_1}}{(k'^2 + d_1 d_2 d_3 d_4)(s_1 s_2 - s_3 s_4)},$$

or

$$s\,(s_1 s_2 - s_3 s_4) = \frac{d_1 d_2 d_3 d_4}{k'^2 + d_1 d_2 d_3 d_4} \Sigma \frac{s_1 c_1}{d_1},$$

and so for others. These, when substituted, lead to expressions already obtained.

8. Passing now to equation (4), which gives the sum of four third elliptic integrals after Jacobi, we consider the fraction whose logarithm occurs in the equation, and for shortness denote it by Ω, so that equation (4) may be written

$$\Sigma \Pi (u_1, a) = \tfrac{1}{2} \log \Omega.$$

The denominator of Ω, on being multiplied out, is found to be

$$= \Delta\delta^2 + k^2\sigma^2\Delta_2 - k^2\sigma^2\delta^2\Delta_3 - 8k^2\sigma\gamma\delta ABC$$

$$= \Delta\left\{\delta^2 + k^2\sigma^2 c_1c_2c_3c_4 - k^2\sigma^2\delta^2 s_1s_2s_3s_4 + \sigma\gamma\delta\Sigma E(u_1)\right\},$$

and the numerator similarly is found to be

$$= \Delta\left\{\delta^2 + k^2\sigma^2 c_1c_2c_3c_4 - k^2\sigma^2\delta^2 s_1s_2s_3s_4 - \sigma\gamma\delta\Sigma E(u_1)\right\}.$$

Hence

$$\Sigma\Pi(u_1, a) = \tfrac{1}{2}\log\left\{\frac{\delta^2 + k^2\sigma^2 c_1c_2c_3c_4 - k^2\sigma^2\delta^2 s_1s_2s_3s_4 - \sigma\gamma\delta\Sigma E(u_1)}{\delta^2 + k^2\sigma^2 c_1c_2c_3c_4 - k^2\sigma^2\delta^2 s_1s_2s_3s_4 + \sigma\gamma\delta\Sigma E(u_1)}\right\},$$

to which various forms can be given by the substitution of the expressions obtained for $\Sigma E(u_1)$.

9. Further, we have

$$\Pi(As \pm Bc \pm Cd - 1) = \Delta(s - s_1)(s - s_2)(s - s_3)(s - s_4),$$

the product on the left-hand side being taken for the four combinations of signs; and, therefore, also

$$\Pi(As \pm Bc \pm Cd + 1) = \Delta(s + s_1)(s + s_2)(s + s_3)(s + s_4).$$

When in these we write $s = (k\sigma)^{-1}$ they respectively give

$$\{(A - k\sigma)^2 + (B\delta + Ck\gamma)^2\}\{(A - k\sigma)^2 + (B\delta - Ck\gamma)^2\}$$
$$= \Delta(1 - k\sigma s_1)(1 - k\sigma s_2)(1 - k\sigma s_3)(1 - k\sigma s_4),$$

and

$$\{(A + k\sigma)^2 + (B\delta + Ck\gamma)^2\}\{(A + k\sigma)^2 + (B\delta - Ck\gamma)^2\}$$
$$= \Delta(1 + k\sigma s_1)(1 + k\sigma s_2)(1 + k\sigma s_3)(1 + k\sigma s_4).$$

The product of the numerator and the denominator of Ω in its original form is therefore

$$\Delta^2(1 - k^2\sigma^2 s_1^2)(1 - k^2\sigma^2 s_2^2)(1 - k^2\sigma^2 s_3^2)(1 - k^2\sigma^2 s_4^2).$$

But this product is also

$$\Delta^2\left[(\delta^2 + k^2\sigma^2 c_1c_2c_3c_4 - k^2\sigma^2\delta^2 s_1s_2s_3s_4)^2 - \sigma^2\gamma^2\delta^2\{\Sigma E(u_1)\}^2\right];$$

hence

$$(\delta^2 + k^2\sigma^2 c_1c_2c_3c_4 - k^2\sigma^2\delta^2 s_1s_2s_3s_4)^2 - \sigma^2\gamma^2\delta^2\{\Sigma E(u_1)\}^2$$
$$= (1 - k^2\sigma^2 s_1^2)(1 - k^2\sigma^2 s_2^2)(1 - k^2\sigma^2 s_3^2)(1 - k^2\sigma^2 s_4^2).$$

This is true for all values of the argument a; the coefficients of the different powers of σ^2 on the two sides must therefore be equal. Hence

$$\{\Sigma E(u_1)\}^2 = k^2\Sigma s_1^2 - 2k^2(1 - c_1c_2c_3c_4 + s_1s_2s_3s_4),$$

$$(1+k^2)\{\Sigma E(u_1)\}^2 = k^4\Sigma s_1^2 s_2^2 - 2k^4 s_1s_2s_3s_4 - k^4(1 - c_1c_2c_3c_4 + s_1s_2s_3s_4)^2,$$

$$\{\Sigma E(u_1)\}^2 = k^4\Sigma s_1^2 s_2^2 s_3^2 - 2k^4 s_1s_2s_3s_4(1 - c_1c_2c_3c_4 + s_1s_2s_3s_4)^2.$$

These equations can be immediately verified when we write $u_4 = 0$.

10. The following formulæ are derivable by starting from the addition-theorem for the third elliptic integral as given by Jacobi (*Fund. Nova*, § 55). In them S_μ, C_μ, D_μ are the elliptic functions of $u_\mu + a$, and S'_μ, C'_μ, D'_μ those of $u_\mu - a$.

$$\Sigma\Pi(u_1, a) = \tfrac{1}{4}\log\Omega',$$

where

$$\Omega' = \frac{S_1S_2C_3C_4D_3D_4 + S_3S_4C_1C_2D_1D_2}{S'_1S'_2C'_3C'_4D'_3D'_4 + S'_3S'_4C'_1C'_2D'_1D'_2}$$

$$= \frac{S_1S_3C_2C_4D_2D_4 + S_2S_4C_1C_3D_1D_3}{S'_1S'_3C'_2C'_4D'_2D'_4 + S'_2S'_4C'_1C'_3D'_1D'_3}$$

$$= \frac{S_1S_4C_2C_3D_2D_3 + S_2S_3C_1C_4D_1D_4}{S'_1S'_4C'_2C'_3D'_2D'_3 + S'_2S'_3C'_1C'_4D'_1D'_4}.$$

There are three values of Ω of the form

$$\frac{D_1D_2S_3S_4C_3C_4 + D_3D_4S_1S_2C_1C_2}{D'_1D'_2S'_3S'_4C'_3C'_4 + D'_3D'_4S'_1S'_2C'_1C'_2},$$

and there are three of the form

$$\frac{C_1C_2S_3S_4D_3D_4 + C_3C_4S_1S_2D_1D_2}{C'_1C'_2S'_3S'_4D'_3D'_4 + C'_3C'_4S'_1S'_2D'_1D'_2}.$$

11. Another example of Abel's Theorem leading to most of the foregoing results arises when we consider as the permanent equation

$$y^2 - (1-x^2)(1-k^2x^2) = 0,$$

and take as the parametric equation

$$y = a + bx + cx^2.$$

The equation in x is of the fourth degree; if we write $x = \operatorname{sn} u$ we have

$$u_1 + u_2 + u_3 + u_4 = 0,$$

and the algebraical equation, which is equivalent to this transcendental relation, is

$$\begin{vmatrix} s_1^2, & s_1, & c_1d_1, & 1 \\ s_2^2, & s_2, & c_2d_2, & 1 \\ s_3^2, & s_3, & c_3d\,, & 1 \\ s_4^2, & s_4, & c_4d_4, & 1 \end{vmatrix} = 0.$$

From the equation which has s_1, s_2, s_3, s_4 for its roots we at once find

$$c\Sigma s_1s_2s_3 = a\Sigma s_1,$$

so that

$$s_4\{c(s_1s_2 + s_2s_3 + s_3s_1) - a\} = a(s_1 + s_2 + s_3) - cs_1s_2s_3.$$

Since the parametric equation is satisfied for $x = s_1, s_2, s_3$, we have, on solving for a, b, c the three equations which arise when these values of x are substituted,

$$-a(s_2 - s_3)(s_3 - s_1)(s_1 - s_2) = c_1d_1s_2s_3(s_2 - s_3) + \ldots,$$

$$-c(s_2 - s_3)(s_3 - s_1)(s_1 - s_2) = c_1d_1(s_2 - s_3) + \ldots.$$

When these values are substituted in the equation which gives s_4, it becomes

$$s_4 = s_1s_2s_3\frac{\dfrac{c_1d_1}{s_1}(s_3^2 - s_2^2) + \dfrac{c_2d_2}{s_2}(s_1^2 - s_3^2) + \dfrac{c_3d_3}{s_3}(s_2^2 - s_1^2)}{s_1c_1d_1(s_3^2 - s_2^2) + s_2c_2d_2(s_1^2 - s_3^2) + s_3c\,d_3(s_2^2 - s_1^2)}.*$$

And it is not difficult to prove that

$$\Sigma E(u_1) = \frac{2bk^2}{c^2 - k^2},$$

$$\Sigma\Pi(u_1, a) = \tfrac{1}{2}\log\frac{(c + ak^2\sigma^2)^2 - k^2(b\sigma - \gamma\delta)^2}{(c + ak^2\sigma^2)^2 - k^2(b\sigma + \gamma\delta)^2}.$$

If the parametric equation be taken to be

$$(b_0 + b_1x + \ldots + b_{n-2}x^{n-2})\,y = a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + a_nx^n,$$

and the permanent equation be the same, there are $2n$ arguments u, such that their sum is zero; and

$$\sum_{\mu=1}^{\mu=2n} E(u_\mu) = 2k^2\,\frac{a_{n-1}b_{n-2} - a_nb_{n-3}}{a_n^2 - k^2b^2_{n-2}}.$$

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* *Messenger of Mathematics*, vol. XIV. p. 25.

AN ALGEBRAICAL TRANSFORMATION.

By Professor *Cayley*.

The following algebraical transformation occurs in a paper by Hermite "On the theory of the Modular Equations," *Comptes Rendus*, t. 48 (1859), p. 1100.

Writing $q=1-2u^8$, $l=1-2v^8$, then in the transformation of the fifth order the modular equation was expressed by Jacobi in the form

$$\Omega, =(q-l)^6-256(1-q^2)(1-l^2)\{16ql(9-ql)^2+9(45-ql)(q-l)^2\}, =0;$$

and if we write herein $q=1-2x$, $l=\dfrac{x+1}{x-1}$ (or, what is the same thing, establish between q, l the relation $q-l=3+ql$,) that is, between u, v the relation $v^8=1\div(1-u^8)$, then the function Ω becomes

$$\Omega=\frac{64}{(1-x)^6}\{(x^2-x+1)^3+2^7(x^2-x)^2\}\{(x^2-x+1)^3+2^7.3^3(x^2-x)^2\};$$

or, what is the same thing, the equation $\Omega=0$ gives for $\dfrac{(x^2-x+1)^3}{(x^2-x)^2}$ the values -2^7 and $-2^7.3^3$.

We, in fact, have

$$q-l=3+ql=\frac{2(x^2-x+1)}{1-x},$$

$$1-q^2=4x(1-x),\quad 1-l^2=\frac{-4x}{(1-x)^2},$$

and therefore

$$(1-q^2)(1-l^2)=\frac{-16x^2}{1-x}.$$

Hence

$$\Omega=\frac{64}{(1-x)^6}\Big[(x^2-x+1)^6+64(1-x)^5$$
$$\times x^2\{16ql(9-ql)^2+9(45-ql)(3+ql)^2\}\Big];$$

and, putting for a moment $ql=\theta-3$, the term in { } is found to be

$$=7\theta^3+3456\theta-6912;$$

viz. this is

$$= \frac{56(x^2-x+1)^2}{(1-x)^3} + \frac{6912(x^2-x+1)}{1-x} - 6912,$$

$$= \frac{8}{(1-x)^3}\{7(x^2-x+1)^3 + 864(x-1)^2(x^2-x+1) + 864(x-1)^3\},$$

$$= \frac{8}{(1-x)^3}\{7(x^2-x+1)^3 + 864(x^2-x)^2\}.$$

Hence

$$\Omega = \frac{64}{(1-x)^6}[(x^2-x+1)^6 + 512(x^2-x)^2\{7(x^2-x+1)^3 + 864(x^2-x)^2\}],$$

which is

$$= \frac{64}{(1-x)^6}\{(x^2-x+1)^3 + 2^7(x^2-x)^2\}\{(x^2-x+1)^3 + 2^7.3^3(x^2-x)^2\}.$$

SOLUTION OF $(a, b, c, d) = (a^2, b^2, c^2, d^2)$.

By Professor *Cayley*.

It is required to find four quantities (no one of them zero) which are in some order or other equal to their squares,

$$(a, b, c, d) = (a^2, b^2, c^2, d^2).$$

Supposing that the required quantities (a, b, c, d) are the roots of the biquadratic equation

$$x^4 + px^3 + qx^2 + rx + s = 0 \qquad (s \text{ not} = 0),$$

then the function

$(x^4 + qx^2 + s)^2 - (px^3 + rx)^2$ must be $= x^8 + px^6 + qx^4 + rx^2 + s$,

and we have thus the conditions

$$2q - p^2 = p, \quad 2s + q^2 - 2pr = q, \quad 2qs - r^2 = r, \quad s^2 = s,$$

the last of which (since s is not $= 0$) gives $s = 1$; and the others then become

$$2q = p^2 + p, \quad 2(pr - 1) = q^2 - q, \quad 2q = r^2 + r;$$

viz., regarding p, q, r as the coordinates of a point in space, this is determined as the intersection of three quadric surfaces, and the number of solutions is thus $= 8$.

We in fact have $2q = p^2 + p = r^2 + r$; that is $p^2 + p = r^2 + r$, or $(p - r)(p + r + 1) = 0$; hence $r = p$ or $r = -1 - p$.

First, if $r = p$; here $2q = p^2 + p$, $2(p^2 - 1) = q^2 - q$: the last equation multiplied by 4 gives

$$8(p^2 - 1) = (p^2 + p)(p^2 + p - 2), = p(p^2 - 1)(p + 2),$$

that is $p^2 - 1 = 0$ or $p^2 + 2p - 8 = 0$.

If $p^2 - 1 = 0$, then either $p = 1$, giving $q = 1$, $r = 1$, and hence the equation $x^4 + x^3 + x^2 + x + 1 = 0$; or else $p = -1$, giving $q = 0$, $r = -1$, and hence the equation $x^4 - x^3 - x + 1 = 0$, that is $(x - 1)^2(x^2 + x + 1) = 0$.

If $p^2 + 2p - 8 = 0$, then either $p = 2$, giving $q = 3$, $r = 2$, and hence the equation $x^4 + 2x^3 + 3x^2 + 2x + 1$, that is

$$(x^2 + x + 1)^2 = 0;$$

or else $p = -4$, giving $q = 6$, $r = -4$, and hence the equation $x^4 - 4x^3 + 6x^2 - 4x + 1 = 0$, that is $(x - 1)^4 = 0$.

Secondly, if $r = -1 - p$; here

$$2q = p^2 + p,\ 2(-p^2 - p - 1) = q^2 - q;$$

the last equation multiplied by 4 gives

$$8(-p^2 - p - 1) = (p^2 + p)(p^2 + p - 2),$$

that is $p^4 + 2p^3 + 7p^2 + 6p + 8 = 0$, or $(p^2 + p + 4)(p^2 + p + 2) = 0$.

If $p^2 + p + 4 = 0$ then $p = \frac{1}{2} - 1 \pm i\sqrt{(15)}$, whence

$$r = \tfrac{1}{2} - 1 \pm i\sqrt{(15)},\quad 2q = p^2 + p,\ = -4,\ \text{or } q = -2;$$

and the equation is

$$x^4 + \tfrac{1}{2}\{-1 \pm i\sqrt{(15)}\}x^3 - 2x^2 + \tfrac{1}{2}\{-1 \pm i\sqrt{(15)}\}x + 1 = 0.$$

If $p^2 + p + 2 = 0$, then $p = \frac{1}{2}\{-1 \pm i\sqrt{(7)}\}$; whence

$$r = \tfrac{1}{2}(-1 \pm i\sqrt{(7)},\quad 2q = p^2 + p,\ = -2,\ \text{or } q = -1;$$

and the equation is

$$x^4 + \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}x^3 - x^2 + \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}x + 1 = 0,$$

that is $(x - 1)[x^3 + \frac{1}{2}\{1 \pm i\sqrt{(7)}\}x^2 + \frac{1}{2}\{-1 \pm i\sqrt{(7)}x - 1] = 0$.

We thus see that the eight equations are

1 $\quad (x - 1)^4 = 0,$

1 $\quad (x^2 + x + 1)^2 = 0,$

1 $\quad (x - 1)^2(x^2 + x + 1) = 0,$

1 $\quad x^4 + x^3 + x^2 + x + 1 = 0,$

2 $\quad (x - 1)\{x^3 + \frac{1}{2}(1 \pm i\sqrt{7})x^2 + \frac{1}{2}(-1 \pm i\sqrt{7})x - 1\} = 0,$

2 $\quad x^4 + \frac{1}{2}(-1 \pm i\sqrt{15})x^3 - 2x^2 + \frac{1}{2}(-1 \mp i\sqrt{15})x + 1 = 0,$

—

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and it hence appears that writing γ, ε, θ to denote respectively an imaginary cube root, fifth root, and seventh root of unity, then that the values of (a, b, c, d) are

$$\begin{array}{cccc}
1, & 1, & 1, & 1; \\
\gamma, & \gamma, & \gamma^2, & \gamma^2; \\
1, & 1, & \gamma, & \gamma^2; \\
\varepsilon, & \varepsilon^2, & \varepsilon^3, & \varepsilon^4; \\
\varepsilon\gamma, & \varepsilon^2\gamma^2, & \varepsilon^4\gamma, & \varepsilon^3\gamma^2; \\
\varepsilon^2\gamma, & \varepsilon^4\gamma^2, & \varepsilon^3\gamma, & \varepsilon\gamma^2; \\
1, & \theta, & \theta^2, & \theta^4; \\
1, & \theta^3, & \theta^6, & \theta^5;
\end{array}$$

viz., for each of these systems we have the required relation

$$(a, b, c, d) = (a^2, b^2, c^2, d^2).$$

It may be noticed that out of the eight equations we have the following three which are irreducible

$$x^4 + x^3 + x^2 + x + 1 = 0,$$

$$x^4 + \tfrac{1}{2}(-1 + i\sqrt{15})\,x^3 - 2x^2 + \tfrac{1}{2}(-1 - i\sqrt{15})\,x + 1 = 0,$$

$$x^4 + \tfrac{1}{2}(-1 - i\sqrt{15})\,x^3 - 2x^2 + \tfrac{1}{2}(-1 + i\sqrt{15})\,x + 1 = 0.$$

Each of these is an Abelian equation, viz. the roots are of the form

$$a,\ \theta(a),\ \theta^2(a),\ \theta^3(a)\ (= a,\ a^2,\ a^4,\ a^8),$$

where $\theta^4(a) = a$, not identically but in virtue of the value of a, viz. we have $\theta^4(a) = a^{16} = a$, in virtue of $a^{15} = 1$: (in the first equation $a^5 = 1$, and therefore $a^{15} = 1$; in each of the other two a^{15} is the lowest power which is $= 1$).

In the first equation we have evidently

$$x^4 + x^3 + x^2 + x + 1$$

as the irreducible factor of $x^5 - 1$.

The second and third equations combined together give

$$(x^4 - \tfrac{1}{2}x^3 - 2x^2 - \tfrac{1}{2}x + 1)^2 + \tfrac{15}{4}(x^3 - x)^2 = 0;$$

that is $$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1 = 0,$$

where the left-hand side is the irreducible factor of $x^{15} - 1$.

NOTE ON A CUBIC EQUATION.

By Prof. *Cayley*.

Consider the cubic equation

$$x^3 + 3cx + d = 0,$$

then effecting upon this the Tschirnhausen-Hermite transformation

$$y = xT_1 + (x^2 + 2c)\,T_2;$$

the resulting equation in y is

$$y^3 + 3y\,(cT_1^2 + dT_1T_2 - c^2T_2^2$$
$$+ dT_2^3 - 6c^2T_1^2T_2 - 3cdT_1T_2^2 - (d^2 + 2c^3)\,T_2^3 = 0,$$

and this will be

$$y^3 + 3cy + d = 0,$$

if only

$$c = cT_1^2 + dT_1T_2 - c^2T_2^2,$$
$$d = dT_1^3 - 6c^2T_1^2T_2 - 3cdT_1T_2^2 - (d^2 + 2c^3)\,T_2^3,$$

equations which give

$$(d^2 + 4c^3) = (d^2 + 4c^3)\,(T_1^3 + 3cT_1T_2^2 + dT_2^3)^2,$$

viz., assuming that $d^2 + 4c^3$ not $= 0$, this is

$$1 = T_1^3 + 3cT_1T_2^2 + dT_2^3.$$

Hence the coefficients T_1, T_2 being such as to satisfy these relations, the equation in z is identical with the equation in x; or what is the same thing if α, β, γ are the roots of the equation in x, then we have beween these roots the relations

$$\beta = \alpha T_1 + (\alpha^2 + 2c)\,T_2,$$
$$\gamma = \beta T_1 + (\beta^2 + 2c)\,T_2,$$
$$\alpha = \gamma T_1 + (\gamma^2 + 2c)\,T_2,$$

viz., the general cubic equation $x^3 + 3cx + d = 0$, adjoining thereto the radicals T_1, T_2 may be regarded as an Abelian equation.

In particular, if $c, d = -1, 1$, then we may write $T_1 = 0$, $T_2 = 1$; the cubic equation is here

$$x^3 + 3x - 1 = 0,$$

and the roots α, β, γ are such that $\beta=\alpha^2-2$, $\gamma=\beta^2-2$, $\alpha=\gamma^2-2$; in fact, taking θ a primitive ninth root of unity, $\theta^6+\theta^3+1=0$; we have $\alpha, \beta, \gamma=\theta+\theta^8, \theta^2+\theta^7, \theta^4+\theta^5$; values which satisfy $x^3+3x-1=0$, and the relations in question.

The same question may be considered from a different point of view: take the transforming equation to be

$$y=A+Bx+Bx^2,$$

then assuming that the values of y corresponding to the values $x=\alpha, \beta, \gamma$ are β, γ, α respectively, we have

$$\beta=A+B\alpha+C\alpha^2,$$

$$\gamma=A+B\beta+C\beta^2,$$

$$\alpha=A+B\gamma+C\gamma^2,$$

and the transforming equation thus is

$$\begin{vmatrix} y, & 1, & x, & x^2 \\ \beta, & 1, & \alpha, & \alpha^2 \\ \gamma, & 1, & \beta, & \beta^2 \\ \alpha, & 1, & \gamma, & \gamma^2 \end{vmatrix}=0:$$

This may also be written

$$(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\{y-\tfrac{1}{2}(\alpha+\beta+\gamma+x\}$$

$$=\ \beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2-\tfrac{1}{2}(\beta^3\gamma+\beta\gamma^3+\gamma^3\alpha+\gamma\alpha^3+\alpha^3\beta+\alpha\beta^3)$$

$$+x\,\{\alpha^3+\beta^3+\gamma^3-\tfrac{1}{2}(\beta^2\gamma+\beta\gamma^2+\gamma^2\alpha+\gamma\alpha^2+\alpha^2\beta+\alpha\beta^2)\}$$

$$+x^2\{\beta\gamma+\gamma\alpha+\alpha\beta-(\alpha^2+\beta^2+\gamma^2)\}.$$

We have

$$(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=\frac{-27}{a^4}(a^2d^2+4ac^3+4b^3d-3b^2c^2-6abcd),$$

$$=\frac{-27}{a^4}\Delta$$

or say $\quad(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)=\dfrac{3(\omega+\omega^2)}{a^2}\sqrt{(\Delta)},$

if Δ be the discriminant, and ω an imaginary cube root of unity, $\{(\omega-\omega^2)^2=-3)$.

The remaining functions of α, β, γ are of course expressible rationally in terms of the coefficients: we have

$$\Sigma\beta^2\gamma^2 = \frac{1}{a^2}(-6bd+9c^2),$$

$$\Sigma\beta^3\gamma = \frac{1}{a^3}(-3abd-18ac^2+27b^2c),$$

$$\Sigma\alpha^3 = \frac{1}{a^3}(-3a^2d+27abc-27b^3),$$

$$\Sigma\beta^2\gamma = \frac{1}{a^2}(3ad-9bc),$$

$$\Sigma\beta\gamma = \frac{3c}{a},$$

$$\Sigma\alpha^2 = \frac{1}{a^2}(9b^2-6ac),$$

and the final result is

$$\begin{aligned}\tfrac{1}{3}(\omega-\omega^2)\sqrt{(\nabla)}\,\{2a(y+x)+3b\} = \quad & -abd+4ac^2-3b^2c \\ & +x\,(-a^2d+7abc-6b^3, \\ & +x^2(\;2a^2c-2ab^2);\end{aligned}$$

viz., we have thus an automorphic transformation of the equation $ax^3+3bx^2+3cx+d=0$.

CORRECTION OF AN ERROR IN A PREVIOUS PAPER.

By *A. R. Forsyth.*

Mr. Robert Rawson has kindly pointed out to me an error which occurs on p. 47 in the note (pp. 44–48 of this volume) on a method of solution of linear differential equations; the integral in line 11, p. 47, should be

$$ve^{\int P_1dx} = B + A\int e^{2\int P_1dx}.$$

The three integrals on p. 48 should be changed respectively to

(i) $yx^{\gamma-1}(1-x)^{\alpha+1-\gamma} = B + A\int x^{\gamma-2}(1-x)^{\alpha-\gamma}dx,$

(ii) $y(1-x)^{\alpha} = B + A\int x^{-\gamma}(1-x)^{\alpha-1}dx,$

(iii) $yx^{\gamma-1} = B + A\int x^{\gamma-2}(1-x)^{-\alpha}dx.$

NOTE ON RATIONALISATION.

By *Capt. P. A. MacMahon, R.A.*

To rationalise the equation

$$a_1^{\frac{1}{m}}+a_2^{\frac{1}{m}}+a_3^{\frac{1}{m}}+\ldots+a_n^{\frac{1}{m}}=0,$$

is the same problem as that of finding the simplest rational algebraic function of the n quantities, which contains the sinister of the above equation as a factor. The case, $m=2$, was considered by Meyer Hirsch at the beginning of this century, and it is given in his algebra; the general case may also have been worked out, but I have thought that its expression, by means of partitions, may not be without interest.

To rationalize the equation

$$y=a_1^{\frac{1}{m}}+a_2^{\frac{1}{m}}+\ldots+a_n^{\frac{1}{m}},$$

is merely the formation of the equation in y, of which this is one solution; this equation will be of degree m^n in y, since the dexter may assume m^n different values; and further, the powers of y will all be of the form m^n-km, k a positive integer, and we may thus put $y^m=x$, obtaining an equation in x wherein the indices decrease by unity, and which will be the rationalization of the equation

$$x^{\frac{1}{m}}=a_1^{\frac{1}{m}}+a_2^{\frac{1}{m}}+\ldots+a_n^{\frac{1}{m}}.$$

The equation will then be of degree m^{n-1}, its roots being the m^{th} powers of the m^{n-1} internally different values of the expression

$$a_1^{\frac{1}{m}}+a_2^{\frac{1}{m}}+\ldots+a_n^{\frac{1}{m}}.$$

If ω_m be an m^{th} root of unity, ω_m will not be the quotient of any two such values.

Let S_r be the sum of the r^{th} powers of the roots of the resulting equation, and let the symmetric function $\Sigma a_1^p a_2^q a_3^r\ldots$ be denoted by the partition symbol $[pqr\ldots]$.

Then it may be shewn that

$$m^{1-n}S_1=[1],$$

$$m^{1-n}S_2=[2]+\frac{(2m)!}{(m!)^2}[1^2],$$

$$m^{1-n}S_3=[3]+\frac{(3m)!}{(2m)!\,m!}[21]+\frac{(3m)!}{(m!)^3}[1^3],$$

$$m^{1-n}S_4=[4]+\frac{(4m)!}{(3m)!\,m!}[31]+\frac{(4m)!}{(2m!)^2}[2^2],$$

$$+\frac{(4m)!}{(2m)!\,(m!)^2}[21^2]+\frac{(4m)!}{(m!)^4}[1^4],$$

....

This is, in fact, merely an interesting extension of the multinomial theorem of algebra which I have not before seen mentioned, though it may possibly have been long known.

The simple multinomial theorem may be expressed thus:—

$$(a_1+a_2+a_3+\ldots+a_n)^r=\Sigma\frac{(l\lambda+m\mu+n\nu+\ldots)!}{(\lambda!)^l\,(\mu!)^m\,(\nu!)^n\ldots}[\lambda^l\mu^m\nu^n\ldots].$$

The generalisation is

$$\Sigma\,(a_1^{\frac{1}{m}}+a_2^{\frac{1}{m}}+\ldots+a_n^{\frac{1}{m}})^{mr}$$

$$=m^{n-1}\Sigma\frac{\{m\,(l\lambda+m\mu+n\nu+\ldots)\}!}{\{(m\lambda)!\}^l\,\{(m\mu)!\}^m\,\{(m\nu)!\}^n\ldots}[\lambda^l\mu^m\nu^n\ldots]$$

where the summation on the sinister side has reference to the internally different values.

For $m=1$, the latter degrades to the former.

Whence $$m^{1-n}S_r$$

$$=\Sigma\frac{\{m\,(l\lambda+m\mu+n\nu+\ldots)\}!}{\{(m\lambda)!\}^l\,\{(m\mu)!\}^m\,\{(m\nu)!\}^n\ldots}[\lambda^l\mu^m\nu^n\ldots].$$

If the equation in x be

$$x^{m^{n-1}}-A_1x^{m^{n-1}-1}+A_2x^{m^{n-1}-2}-\ldots=0,$$

we know from the general theory of symmetric functions that

$$A_j=\Sigma\frac{\left(\frac{S_1}{1}\right)^{t_1}\left(\frac{S_2}{2}\right)^{t_2}\ldots\left(\frac{S_j}{j}\right)^{t_j}}{t_1!\;t_2!\;\ldots t_j!}(-)^{j+\Sigma t},$$

wherein $$t_1 + 2t_2 + 3t_3 + \ldots + jt_j = j.$$

We have thus the means of calculating the coefficients in succession.

The term not involving x when equated to zero will be the m^{th} power of the rationalisation of

$$a_1^{\frac{1}{m}} + a_2^{\frac{1}{m}} + \ldots + a_n^{\frac{1}{m}} = 0,$$

which rationalisation consequently will consist of the continued product of the m^{n-1} internally different values of the sinister term, and its degree will be m^{n-2}.

The same result is arrived at much more easily by putting $x = \pm a_{n+1}$, and then putting $n-1$ for n.

The great advantage of this method is that it is only necessary to calculate about half the coefficients, the known symmetry of the result enabling one to write down the remaining terms.

It will suffice in this short note to give two results arrived at by this process,

$$a_1^{\frac{1}{5}} + a_2^{\frac{1}{5}} + a_3^{\frac{1}{5}} = 0,$$

leads to

$$(a_1+a_2+a_3)^5 - 5^4(a_1+a_2+a_3)^2 a_1a_2a_3 + 5^5(a_2a_3+a_3a_1+a_1a_2)a_1a_2a_3 = 0;$$

and $$a_1^{\frac{1}{7}} + a_2^{\frac{1}{7}} + a_3^{\frac{1}{7}} = 0,$$

leads to

$$(a_1+a_2+a_3)^7 - 5.7^4(a_1+a_2+a_3)^4 a_1a_2a_3$$
$$+ 2.7^6(a_1+a_2+a_3)^2(a_2a_3+a_3a_1+a_1a_2)\,a_1a_2a_3$$
$$- 7^7(a_2a_3+a_3a_1+a_1a_2)^2 a_1a_2a_3 + 7^8(a_1+a_2+a_3)\,a_1^2a_2^2a_3^2 = 0.$$

Royal Military Academy,
Woolwich.
August 24*th*, 1885.

NOTE IN CONNEXION WITH FERMAT'S LAST THEOREM.

By *G. B. Mathews, B.A.*

FERMAT'S famous theorem that the indeterminate equation $z^n = x^n + y^n$, where n is a positive integer, cannot be solved in integers except when $n = 2$, has been demonstrated by Kummer, with the exception of certain special cases, by means of his theory of ideal primes. This method, however, is decidedly complicated; moreover, Fermat asserted that he had discovered a proof, and there does not seem to be any sufficient reason for doubting his veracity. It is very improbable that Fermat's method had anything to do with the ideal theory; so that there remains a possibility of discovering or re-discovering an independent proof of the theorem. The following note is published in the hope that it may suggest a method of solving the problem, or at any rate of simplifying it, by means of ordinary congruences.

The method adopted was suggested by Gauss's article "Neue Theorie der Zerlegung der Cuben," (*Werke*, II., pp. 387–391).

Fermat has left a proof of the theorem when $n = 4$, so that to prove it in general, it is sufficient to consider the equation $z^p = x^p + y^p$, where p is an uneven prime. The first part of Kummer's demonstration consists in shewing the impossibility of solving the equation if none of the integers x, y, z is a multiple of p. All that is accomplished by the following analysis is the simplification of this part of the process for a few special values of p. It is assumed throughout that x, y, z have no common divisor; and all the congruences, except when another modulus is expressly mentioned, are relative to p.

By Fermat's theorem

$$x^p \equiv x, \quad y^p \equiv y, \quad z^p \equiv z,$$

so that if

$$z^p = x^p + y^p,$$

$$z \equiv x + y.$$

Assume, therefore,

$$z = pt + x + y,$$

then

$$(pt + x + y)^p = x^p + y^p,$$

whence

$$(x + y)^p - x^p - y^p \equiv 0 \pmod{p^2}.$$

Now $$(x + y)^p - x^p - y^p = pxy(x + y)\,\phi(x, y),$$

where $\phi(x, y)$ is a homogeneous integral function of degree $(p-3)$, thus the congruence last written is equivalent to

$$xy(x+y)\,\phi(x, y) \equiv 0 \pmod{p},$$

or, since $$z \equiv x + y,$$

to $$xyz \,.\, \phi(x, y) \equiv 0.$$

If, therefore, it can be shewn that the congruence $\phi(x, y) \equiv 0$ is insoluble, unless one or more of the quantities x, y, $x+y$ is divisible by p, it will follow that one of the three x, y, z must be a multiple of p.

Considering the simplest cases in order, we have

I. $p = 3$,

$$(x+y)^3 - x^3 - y^3 = 3xy(x+y).$$

Here ϕ is a constant, and the theorem is true.

II. $p = 5$,

$$(x+y)^5 - x^5 - y^5 = 5xy(x+y)(x^2 + xy + y^2),$$

the congruence $$x^2 + xy + y^2 \equiv 0$$

gives $$4(x^2 + xy + y^2) \equiv 0,$$

or $$(2x+y)^2 + 3y^2 \equiv 0.$$

Now -3 is a non-residue of 5, so that the only possible solution is $y \equiv 0$, $2x + y \equiv 0$, whence also $x \equiv 0$ and $z \equiv 0$, contrary to the supposition that x, y, z have no common divisor. In this case, therefore, the theorem is true.

III. $p = 7$,

$$(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2 + xy + y^2)^2,$$

so that $$xyz \equiv 0,$$

or else $$x^2 + xy + y^2 \equiv 0;$$

-3 being a residue of 7, the latter congruence is soluble, *e.q.* by putting $x = 2$, $y = 1$, and the truth of the theorem remains doubtful.

IV. $p = 11$,

$$\phi = (x^2 + xy + y^2)(x^6 + 3x^5y + 7x^4y^2 + 9x^3y^3 + 7x^2y^4 + 3xy^5 + y^6),$$

-3 is a non-residue of 11, so that the factor $x^2 + xy + y^2$ may be rejected; and the second factor of ϕ

$$\begin{aligned}
&\equiv x^6 - 8x^5y + 29x^4y^2 - 24x^3y^3 + 29x^2y^4 - 8xy^5 + y^6 \\
&\equiv (x^3 - 4x^2y + 4xy^2 - y^3)^2 + 5x^2y^2(x+y)^2 \\
&\equiv X^2 + 5Y^2, \text{ say;}
\end{aligned}$$

-5 is a non-residue of 11, and the only solution is $X \equiv 0$, $Y \equiv 0$, leading, as before, to $x \equiv y \equiv z \equiv 0$. (Theorem true).

V. $p = 13$,

$$\phi = (x^2 + xy + y^2)^2 (x^6 + 3x^5y + 8x^4y^2 + 11x^3y^3 + 8x^2y^4 + 3xy^5 + y^6),$$

-3 is a residue of 13, and the theorem is doubtful.

VI. $p = 17$,

$$\phi = (1, 7, 33, 107, 257, 471, 673, 757, \ldots)(x, y)^{14},$$

(a reciprocal function of which $757x^7y^7$ is the middle term)

$$= (x^2 + xy + y^2)(1, 6, 26, 75, 156, 240, 277, \ldots)(x, y)^{12}.$$

Substituting for the coefficients their least positive residues, mod. 17,

$$\phi \equiv (x^2 + xy + y^2)(1, 6, 9, 7, 3, 2, 5, \ldots)(x, y)^{12}.$$

The first factor may be rejected, since $-3N17$; the second

$$\equiv (x^6 + 3x^5y - 5x^3y^3 + 3xy^5 + y^6)^2 - x^4y^4(x+y)^4.$$

Hence, if $\phi \equiv 0$,

$$x^6 + 3x^5y - 5x^3y^3 + 3xy^5 + y^6 \equiv \pm x^2y^2(x+y)^2.$$

This may be otherwise written

$$(x^2 + xy + y^2)^3 \equiv (6 \pm 1)\, x^2y^2(x+y)^2;$$

so that we have to consider the congruences

$$\xi^3 \equiv 5\eta^2 \text{ and } \xi^3 \equiv 7\eta^2,$$

where $\qquad \xi = x^2 + xy + y^2, \quad \eta = xy(x+y).$

It may be shewn by actual trial, that the only solution is $\xi \equiv 0$ and $\eta \equiv 0$, leading as before to $x \equiv y \equiv z \equiv 0$, so that the theorem is proved for this case also.

With regard to the process of trial, it may be observed that since 5 and 7 are both non-residues of 17, ξ is in each

case a non-residue of 17; and it is thus found that all the admissible values of ξ, η are, for the first congruence,

$$\xi \equiv \quad 3, \quad 5, \quad 6, \quad 7, \quad 10, \quad 11, \quad 12, \quad 14,$$
$$\eta \equiv \pm 6, \pm 5, \pm 4, \pm 2, \pm 8, \pm 1, \pm 3, \pm 7;$$

and for the second,

$$\xi \equiv \quad 3, \quad 5, \quad 6, \quad 7, \quad 10, \quad 11, \quad 12, \quad 14,$$
$$\eta \equiv \pm 4, \pm 8, \pm 3, \pm 7, \pm 6, \pm 5, \pm 2, \pm 1.$$

As a specimen of the work, suppose

$$x^2 + xy + y^2 \equiv 3 \quad \text{(i)},$$
$$xy(x+y) \equiv \pm 6 \quad \text{(ii)}.$$

Then
$$(x+y)^2 \equiv 3 + xy,$$
$$x^2y^2(3+xy) \equiv 36,$$
$$x^3y^3 + 3x^2y^2 - 36 \equiv 0;$$
$$(xy-16)(x^2y^2 + 2xy - 2) \equiv 0,$$
or
$$(xy-16)\{(xy+1)^2 - 3\} \equiv 0.$$

Since $3N17$, the only solution is

$$xy \equiv 16 \quad \text{(iii)},$$

adding (iii) to (i),
$$(x+y)^2 \equiv 19 \equiv 2;$$
therefore
$$x + y \equiv \pm 6 \quad \text{(iv)}.$$

Multiplying by x, and using (iii),

$$x^2 - 1 \equiv \pm 6x,$$
whence
$$(x \mp 3)^2 \equiv 10,$$
which is impossible, since $10N17$.

Writing, for the moment, t instead of xy, the remaining cases of $\xi^3 \equiv 5\eta^2$ lead similarly to the congruences

$$t^2(t+5) \equiv 8, \qquad t^2(t+6) \equiv 16, \qquad t^2(t+7) \equiv 4,$$
$$t^2(t+10) \equiv 13, \qquad t^2(t+11) \equiv 1, \qquad t^2(t+12) \equiv 9,$$
$$t^2(t+14) \equiv 15.$$

It will be found that these are equivalent to the following:

$$(t-4)\{(t-4)^2-14\}\equiv 0,$$
$$(t+2)\{(t+2)^2-12\}\equiv 0,$$
$$(t-9)\{(t-9)^2-5\}\equiv 0,$$
$$(t-8)\{(t-8)^2-9\}\equiv 0,$$
$$(t-2)\{(t-2)^2-12\}\equiv 0,$$
$$(t+4)\{(t+4)^2-14\}\equiv 0,$$
$$(t-1)\{(t-1)^2-3\}\equiv 0.$$

Since 3, 5, 12, 14 are all non-residues of 17, it follows that the only real solutions are $t\,(=xy)\equiv 4,\ -2,\ 9,\ 8,\ 2,\ -4,\ 1$ respectively.

Corresponding to these, we have a set of insoluble congruences in x; namely,

$$(x\pm 7)^2\equiv 11,\qquad (x\mp 1)^2\equiv 3,\qquad (x\mp 2)^2\equiv 12,$$
$$(x\pm 8)^2\equiv 5,\qquad (x\pm 4)^2\equiv 14,$$
$$(x\pm 6)^2\equiv 6,\qquad (x\pm 5)^2\equiv 7.$$

In a similar way the congruence $\xi^3\equiv 7\eta^2$ leads to the following

$$t^2(t\pm 3)\equiv\mp 1,\qquad t^2(t\pm 5)\equiv\mp 4,$$
$$t^2(t\pm 6)\equiv\mp 8,\qquad t^2(t\pm 7)\equiv\mp 2,$$

which may be written in the form

$$(t\pm 1)^3\equiv 3t,$$
$$(t\pm 2)^3\equiv 12t,$$
$$(t\pm 4)^3\equiv 14t,$$
$$(t\pm 8)^3\equiv 5t.$$

Now it may be shewn that none of these congruences can be satisfied by a real value of t. Taking them in the first form, and removing the ambiguities, we have the following equivalent set:

$$t^6-(3t^2+1)^2\equiv 0,$$
$$t^6-(5t^2+4)^2\equiv 0,$$
$$t^6-(6t^2+8)^2\equiv 0,$$
$$t^6-(7t^2+2)^2\equiv 0.$$

Reducing the congruences, and multiplying them all together, the resulting congruence is

$$t^{24} + 6t^{16} + 2t^{8} - 1 \equiv 0.$$

Now for all real values of t, $t^8 \equiv 0$ or ± 1, and none of these values will do; hence, all the congruences involved are irreducible, as stated.

The final congruence in t may be transformed by the help of $t^{16} \equiv 1$ into

$$3t^8 + 5 \equiv 0,$$

whence $t^8 - 4 \equiv 0$; and it will be found that this is equivalent to

$$(t^2 - 6)(t^2 - 7)(t^2 - 10)(t^2 - 11) \equiv 0,$$

the insolubility of which is evident.

Unhappily the method of which a sketch has been presented is deficient in two serious respects; first, there is the difficulty of shewing that $z^p = x^p + y^p$ is impossible when z, say, is divisible by p; secondly, there is the consideration of primes p of the form $3n + 1$. In this case ϕ involves a factor $(x^2 + xy + y^2)^2$, and the congruence $x^2 + xy + y^2 \equiv 0$ is soluble (cf. the cases $p = 7$, 13 above); it can be shewn without difficulty that $z = x + y$ (mod p^2), but even when $p = 7$, I have been unable to make any further progress worth mentioning. The congrnence $x^2 + xy + y^2 \equiv 0$ (mod 7) gives $x \equiv 2y$ or $4y$; if the former be adopted, we may write

$$x = 7u + 2y,$$

$$z = 49t + x + y = 49t + 7u + 3y.$$

Substituting in $z^7 = x^7 + y^7$, we have an equation in t, u, y, and from this may be deduced a congruence $ty \equiv (y + 5u)^2$; however, this does not seem to lead to anything.

It may be worth noticing that when $p = 13$,

$$\phi \equiv (x-3y)^2(x-9y)^2(x-2y)(x-4y)(x-5y)(x-7y)(x-8y)(x-10y),$$

so that the congruence

$$\frac{(x+y)^{13} - x^{13} - y^{13}}{13} \equiv 0 \pmod{13}$$

has 13 real roots.

On the other hand, when $p = 19$,

$$\phi = (x^2 + xy + y^2)^2 \,(1, 6, 28, 85, 184, 292, 341 \ldots \between x, y)^{12},$$

and the second factor

$$\equiv (x^6 + 3x^5y - 5x^3y^3 + 3xy^5 + y^6)^2 + 5x^4y^4(x+y)^4.$$

Since $-5N19$ this will $\equiv 0$ only if $xy(x+y) \equiv 0$, and $x^6 + 3x^5y + \ldots \equiv 0$, which involves $x \equiv y \equiv 0$.

University College, Bangor,
Oct. 1885.

NOTE ON SCHWARZIAN DERIVATIVES.

By Professor *Sylvester*.

Reading with great pleasure and profit Mr. Forsyth's masterly treatise on Differential Equations (in my opinion the best written mathematical book extant in the English language), it occurred to me to find an easy proof of the fundamental and striking identity concerning Schwarzian derivatives, from which all others are immediate consequences, viz. $(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 (y, z)$ where one of which is, it may be observed, that (y, x) like y'' has the property of remaining a factor of what it becomes when x and y are interchanged; a persistent factor, so to say, of its altered self. I will return to this point subsequently, my present concern is to give a natural proof of the above striking identity; to do this, it will be sufficient to show that (considering y, z, x, the two former as fixed, and the last as a variable function of a common variable) $\dfrac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}$ does not vary when x becomes $x + \varepsilon(\phi x)$ where ε may be regarded as infinitesimal.* For then this must remain true by successive accumulation when x becomes any function whatever of itself, and accordingly making $x = z$ we obtain (y, z) as the value of the invariable quotient as was to be shown. Call $\varepsilon\delta\phi x = \theta$, then using dashes to denote differentiation *quâ* x, and a parenthesis to signify the augmented value of the derivatives, we obtain

$$(y') = y' - \theta y',$$
$$(y'') = y'' - 2\theta y'' - \theta' y',$$
$$(y''') = y''' - 3\theta y''' - 3\theta' y'' - \theta'' y'.$$

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* It is easy to see *á priori* that if the theorem is true, it can only be so in virtue of (y, x) when x receives an infinitesimal, becoming of the form

$$(1 - 2\theta)(y, x) + \lambda\theta'',$$

as is subsequently shown to be the case in the text.

Hence $$(y'y''') = y'y''' - 4\theta y'y''' - 3\theta' y'y'' - \theta'' y'^2,$$
$$\tfrac{3}{2}(y''^2) = \tfrac{3}{2}y''^2 - 6\theta y''^2 - 3\theta' y'y'',$$
$$(y')^2 = y'^2 - 2\theta y'^2.$$
$$((y, x)) = (1 + 2\theta)\{(y, x) - 4\theta(y, x)\} - \theta''$$
$$= (1 - 2\theta)(y, x) - \theta'',$$
and $$((y, x) - (z, x)) = (1 - 2\theta)[(y, x) - (z, x)].$$
Hence $$\left(\frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2}\right) = \frac{(y, x) - (z, x)}{\left(\frac{dz}{dx}\right)^2},$$
i.e. the right-hand expression does not change, when y, z remaining fixed forms of function, x passes from one form of function of the independent variable to another; as was to be shown.

From what precedes, it appears that if y, z, x be regarded as functions of t, then $\{(y, x) - (z, x)\}\left(\frac{dx}{dt}\right)^2$ is a constant function in the sense that it remains unaltered, whatever function x may be of t, or which is the same thing if y and z functions of x when expressed as functions of x' (any function of x) are written y', z', then $(y', x') - (z', x')$ is identical with $(y, x) - (z, x)$, save as to a *factor* which depends only on the form of the *substitution* of x' for x. Hence to all intents and purposes, any functions of the differences of the Schwarzian derivatives of any system of functions of the same variable, in respect thereto, is (in a sense comprising, but infinitely transcending the sense in which that word is used in Algebra) a *covariant* of the system.

ADDENDUM.—Let us for the moment call functions of x, y which either remain unaltered or only change their sign when x and y are interchanged self-reciprocating functions.

The first case of the kind is $\frac{y''}{y'^{\frac{3}{2}}}$, the next is $\frac{y'y''' - y''^2}{y'^2}$, and obviously a very general one of this sort will be the function
$$\left(\frac{1}{y'^{\frac{1}{2}}}\frac{d}{dx}\right)\log y'.$$

For greater simplicity, let us call the *numerator* of any such function when expanded and brought to the lowest possible common denominator, a *reciprocant*, the highest index of differentiation which such reciprocant contains its *order*, and the number of factors in each term its *degree*. Then in any reciprocant so formed the degree is always just one unit less

than the order: but as a matter of fact the function so obtained is in general not irreducible, so that its degree may be depressed, and it becomes a question of much interest to form the scale of degrees of reciprocants of this sort. For the orders 2, 3, 4, 5, 6 the degrees in question are respectively 1, 2, 2, 3, 3. Calling the successive derivatives of y, a, b, c, d, ..., they will be found to be

$$\begin{array}{l} a, \\ b, \\ 2ac - 3b^2, \\ ad - 5bc, \\ 2a^2e - 15acd - 10ad^2 + 35b^2c, \\ 2a^2f - 21abc - 35acd + 60ab^2d + 110bc^2, \end{array}$$

where each form is obtained by operating upon the preceding one with the operator $a\,(b\delta c + c\delta d + d\delta e + \ldots) - \lambda b$ (λ meaning half the weight + the degree of the operand), combining the result of this operation in each *alternate* case with a *legitimate* combination of those that precede, and in that case dividing out by a. I have proved that in this way can be obtained an infinite progression of reciprocants, of which the leading terms (substituting numbers for letters), will be alternately of the forms $1^i.(2i+1)$ and $1^i.(2i+2)$. Every other reciprocant can be formed algebraically from these primordial forms, as every seminvariant can be obtained from the primordial forms a, $ac-b^2$, $a^2d-3abc+2b^3$, ... The two theories run in parallel courses, but their relationship is that which naturalists call homplazy as distinguished from homogeny; I propose to give further developments of this new algebraical theory in a subsequent Note.

Oxford, *Oct.* 21, 1885.

NOTE ON LINEAR ASSOCIATIVE ALGEBRA.

By *Arthur Buchheim, M.A.*

IN his paper "On Double Algebra" (*Proc. Lond. Math. Soc.* XV. 185), Prof. Cayley says, after quoting the Multiplication Tables for the three double algebras found by Peirce, "to these, however, should be added the system

$$(d_2)\quad \begin{array}{c|c|c} & x & y \\ \hline x & x & 0 \\ \hline y & y & 0 \end{array}$$

see *post*, No. 19," and accordingly on p. 197, Prof. Cayley shows that "the two systems (b_2), (d_2) are distinct systems, in nowise transformable, the one into the other." Now on p. 47 of the original edition of Peirce's "Linear Associative Algebra," he says that "the hypothesis that the other unit belongs to the third group $[d_2]$ is a virtual repetition of [12]," *i.e.* b_2. In this note I attempt to explain this apparent contradiction.

I call to mind that, using Peirce's notation, the multiplication table of (b_2) is

	i	j
i	i	j
j	0	0

And the multiplication table of (d_2) is

	i	j
i	i	0
j	j	0

Now consider the product

$$AB = (xi + yj)\,(x'i + y'j)\,;$$

if we multiply according to (b_2), we get

$$(AB)_1 = xx'i + yx'j\,;$$

if we multiply according to (d_2), we get

$$(AB)_2 = xx'i + xy'j,$$

and it is obvious that $(AB)_1 = (BA)_2$. Therefore the two algebras are virtually identical, inasmuch as every product formed according to (d_2) is identical with a certain product formed according to (b_2).

It will be found that a relation of this kind always holds in the cases in which Peirce rejects an algebra as being virtually a repetition of another. Thus, in the case of triple algebra, Peirce takes an algebra [12], in which j is in the first and k in the second group, and rejects an algebra [14] in which j is in the first and k in the second group, as being virtually a repetition of [12].

For [12] we have

$$i^2 = i, \quad ij = ji = j, \quad j^2 = 0, \quad ik = k, \quad ki = 0,$$

and therefore $\qquad kj = k\,(ij) = 0,$

and $\qquad k^2 = k\,(ik) = (ki)\,k = 0.$

Moreover, if we assume

$$jk = a_{23}i + b_{23}j + c_{23}k,$$

we get, by post-multiplying by i,

$$0 = a_{23}i + b_{23}j,$$

and therefore $\qquad jk = c_{23}k,$*

and since j^2 vanishes

$$c_{23} = 0 = jk.$$

If we use these laws, we find

$$(AB)_1 = (xi + yj + zk)\,(x'i + y'j + z'k)$$
$$= xx'i + (xy' + x'y)\,j + xz'k.$$

Now for [14] we get in the same way

$$i^2 = i, \;\; ij = ji = j, \;\; j^2 = 0, \;\; ki = k, \;\; ik = 0, \;\; k^2 = 0, \; jk = 0, \;\; kj = 0,$$

and we have

$$(AB)_2 = (xi + yj + zk)\,(x'i + y'j + z'k)$$
$$= xx'i + (yx' + xy')\,j + zx'k,$$

and obviously $\qquad (AB)_1 = (BA)_2,$

and we see, as before, that the two algebras are virtually identical, inasmuch as every product formed according to [12] is identical with a certain product formed according to [14].

The Grammar School, Manchester,
Oct. 7, 1885.

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* This also follows from Peirce's general multiplication table for units of different groups.

ON A CERTAIN CLASS OF DEFINITE INTEGRALS.

By *R. Fujisawle.*

THE following expansion may be conveniently used in evaluating a certain class of definite integrals.

Let

$$x+y\cos\theta=(a+be^{i\theta})(a+be^{-i\theta})=a^2+b^2+2ab\cos\theta,$$

where we suppose x and y real and positive, and x greater than y. It follows immediately that $a^2+b^2=x$, $2ab=y$, so that

$$a=\tfrac{1}{2}\{\sqrt{(x+y)}+\sqrt{(x-y)}\},\quad b=\tfrac{1}{2}\{\sqrt{(x+y)}-\sqrt{(x-y)}\},$$

$$\log(x+y\cos\theta)=2\log a+\log\left(1+\frac{b}{a}e^{i\theta}\right)\left(1+\frac{b}{a}e^{-i\theta}\right)$$

$$=2\log a+\frac{b}{a}e^{i\theta}-\tfrac{1}{2}\left(\frac{b}{a}\right)^2e^{2i\theta}+\ldots ad\ inf.$$

$$+\frac{b}{a}e^{-i\theta}-\tfrac{1}{2}\left(\frac{b}{a}\right)^2e^{-2i\theta}+\ldots ad\ inf.$$

$$=2\log a+2\left(\frac{b}{a}\right)\cos\theta-2.\tfrac{1}{2}\left(\frac{b}{a}\right)^2\cos2\theta+\ldots$$

$$-(-1)^n\frac{2}{n}\left(\frac{b}{a}\right)^n\cos n\theta+\ldots ad\ inf.,$$

that is,

$$\log(x+y\cos\theta)=2\log a+2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(\frac{b}{a}\right)^n\cos n\theta\ \ldots\ldots\ (1).$$

Differentiate (1) m times with respect to x, and we obtain

$$(-1)^{m-1}(m-1)!\,\frac{1}{(x+y\cos\theta)^m}=2\left(\frac{d}{dx}\right)^m(\log a)$$

$$+2\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{d}{dx}\right)^m\left(\frac{b}{a}\right)^n\cos n\theta\ \ldots\ldots\ldots\ (2).$$

Now $\dfrac{b}{a}$ is equal to

$$\frac{\sqrt{(x+y)}-\sqrt{(x-y)}}{\sqrt{(x+y)}-\sqrt{(x-y)}}\ \text{and}\ a^2=\tfrac{1}{2}\{x+\sqrt{(x^2-y^2)}\},$$

so that $x > y$ is the necessary and sufficient condition that the right-hand series should converge. We may more conveniently write $\left(\frac{d}{dx}\right)^{m-1} \frac{1}{\sqrt{(x^2-y^2)}}$ instead of $2\left(\frac{d}{dx}\right)^m \log a$, and (2) may be written

$$(-1)^{m-1}(m-1)!\,\frac{1}{(x+y\cos\theta)^m} = \left(\frac{d}{dx}\right)^{m-1} \frac{1}{\sqrt{(x^2-y^2)}}$$

$$+2\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{d}{dx}\right)^m \left[\frac{\sqrt{(x+y)}-\sqrt{(x-y)}}{\sqrt{(x+y)}+\sqrt{(x-y)}}\right] \cos n\theta \ldots (3).$$

Put $m=1$, $x=1+x'^2$, $y=2x'$ where $x'<1$, and we obtain the well known expansion

$$\frac{1-x^2}{1-2x\cos\theta+x^2} = 1+2x\cos\theta+2x^2\cos 2\theta+\ldots.$$

Again $m=2$, $x=1$, $y=\sin 2\lambda$, gives immediately

$$\frac{1}{(1+\sin 2\lambda\cos\theta)^2}$$

$$=\sec^2 2\lambda\,\{1+\sum_{n=1}^{\infty}(-1)^n \tan^n\lambda\,(1+n\cos 2\lambda)\cos n\theta\},\ (\lambda<\tfrac{1}{4}\pi),$$

which is to be found in the Mathematical Tripos Problems of a few years ago.

Multiplying (3) by $d\theta$, and integrating between 0 and π, we have

$$\int_0^\pi \frac{d\theta}{(x+y\cos\theta)^m} = (-1)^{m-1}\frac{1}{(m-1)!}\pi\left(\frac{d}{dx}\right)^{m-1}\frac{1}{\sqrt{(x^2-y^2)}} \ldots (4),$$

Again multiply (3) by $\cos n\theta\,d\theta$, and integrate between 0 and π; we have

$$\int_0^\pi \frac{\cos n\theta\,d\theta}{(x+y\cos\theta)^m}$$

$$=\frac{2\pi}{(m-1)!}(-1)^{m+n}\left(\frac{d}{dx}\right)^m \left[\frac{\sqrt{(x+y)}-\sqrt{(x-y)}}{\sqrt{(x+y)}+\sqrt{(x-y)}}\right]^n \ldots (5);$$

(4) and (5) are only a few of numerous definite integrals which admit of evaluation by means of (3). Moreover, most of the known trigonometrical expansions which are usually given in text books, follow immediately from (3).

Strassburg.

EXPANSION OF A FUNCTION IN A FUNCTIONAL SERIES.

By *A. R. Johnson.*

THE expansion of a function in a trigonometrical series is investigated in the third chapter of Fourier's 'Theory of Heat' from the point of view which considers the identity of the expansions of the series and the function it represents in ascending powers of the argument. The same method may be applied to effect the expansion of a function in a general functional series. It is the object of this article to effect the expansion, to deduce from the several formulæ Fourier's theorems, and to consider the effect of a gap in the otherwise continuous series. The functions are supposed expansible in ascending powers of the argument.

Let $f(x)$ and $\phi(x)$ be rational and integral functions of x, the latter containing every power of x that the former does, and let $a_1, a_2, \ldots a_n$ be any n distinct quantities, $n-1$ being not less than the degree of $\phi(x)$.

We may put

$$f(x) = A_1\phi(a_1x) + A_2\phi(a_2x) + \ldots + A_r\phi(a_rx) + \ldots + A_2\phi(a_nx),$$

where $A_1, A_2 \ldots A_n$ are constants depending on $a_1, a_2 \ldots a_n$. Expanding each side by Maclaurin's Theorem, and equating coefficients of like powers of x, we get

$$A_1 + A_2 + \ldots + A_r + \ldots + A_n = \frac{f(0)}{\phi(0)},$$

$$A_1a_1 + A_2a_2 + \ldots + A_ra_r + \ldots + A_na_n = \frac{f'(0)}{\phi'(0)},$$

$$A_1a_1^2 + A_2a_2^2 + \ldots + A_ra_r^2 + \ldots + A_na_n^2 = \frac{f''(0)}{\phi''(0)},$$

....

$$A_1a_1^{n-1} + A_2a_2^{n-1} + \ldots + A_ra_r^{n-1} + \ldots + A_na_n^{n-1} = \frac{f^{(n-1)}(0)}{\phi^{(n-1)}(0)},$$

equations just sufficient in number to determine the A's.

Let I be such a symbol of operation that $I^r\dfrac{f(0)}{\phi(0)}$, or for shortness $I^r\dfrac{f}{\phi}$, represents $\dfrac{f^{(r)}(0)}{\phi^{(r)}(0)}$. Then I obeys the usual combinational laws of analysis.

Solving for A_r from the above equations, and making use of the expression of the product of the difference of any numbers in a determinantal form, we get

$$A_r = \frac{(I-a_1)(I-a_2)\dots(I-a_{r-1})(I-a_{r+1})\dots(I-a_n)}{(a_r-a_1)(a_r-a_2)\dots(a_r-a_{r-1})(a_r-a_{r+1})\dots(a_r-a_n)}\frac{f}{\phi}.$$

Hence, substituting for A_r,

$$f(x)=\Sigma\phi(a_rx)\frac{(I-a_1)(I-a_2)\dots(I-a_{r-1})(I-a_{r+1})\dots(I-a_n)}{(r_r-a_1)(a_r-a_2)\dots(a_r-a_{r-1})(a_r-a_{r+1})\dots(a_r-a_n)}\frac{f}{\phi}$$
$$\dots\dots\dots\dots\dots\dots(A),$$

where Σ denotes summation with respect to the a's, and the product in I is to be developed and $\dfrac{f^{(r)}(0)}{\phi^{(r)}(0)}$ substituted for $I^r\dfrac{f}{\phi}$.

As an example of the application of (A), take the following:—

In a prism the law of variation of a physical quantity y is given as a partial differential equation in t the time and x the coordinate of position. At the ends $x=0$, $x=\xi$, y vanishes for all values of t, and when $t=0$, the law of distribution of the quantity is $y=f(x)$.

Suppose that we get $y=\psi(a, t)\phi(ax)$ as a possible solution of the differential equation, where a is any constant and $\phi(ax)$ vanishes with x. Then if we take the series $\Sigma A_r\psi(a_r, t)\phi(a_rx)$, where $a_r\xi=\rho_r$ a root of the equation $\phi(x)=0$, and Σ has reference to summation with respect to all the roots, we see that the series is a proper representation of y since it satisfies the differential equation and the terminal conditions. Putting $t=0$, we must determine the coefficients A, by means of the relation

$$f(x)=\Sigma A_r\psi(a_r, 0)\phi(a_rx);$$

but $f(x)=$

$$\Sigma\frac{(I-a_1)(I-a_2)\dots(I-a_{r-1})(I-a_{r+1})\dots(I-a_n)}{(a_r-a_1)(a_r-a_2)\dots(a_r-a_{r-1})(a_r-a_{r+1})\dots(a_r-a_n)}\frac{f}{\phi}\phi(a_rx);$$

and therefore

$$=\Sigma\frac{\phi(a_rx)}{\phi'(\rho_r)}\frac{\phi(I\xi)}{I\xi-\rho_r}\frac{f}{\phi}.$$

Thus

$$A_r=\frac{1}{\phi'(\rho_r)\psi(a_r, 0)}\frac{\phi(I\xi)}{I\xi-\rho_r}\frac{f}{\phi}.$$

The value of y therefore is

$$y=\Sigma\frac{\phi\left(\rho_r\frac{x}{\xi}\right)\psi(a_r,t)}{\phi'(\rho_r)\psi(a_r,0)}\frac{\phi(I\xi)}{I\xi-\rho_r}\frac{f}{\phi}\text{.......................}(\alpha),$$

where the summation is to extend to all values of ρ.

Now, returning to the formula (A) and attaching the values $-\frac{\infty}{\rho}, \ldots, -\frac{2}{\rho}, -\frac{1}{\rho}, 0, +\frac{1}{\rho}, +\frac{2}{\rho}, \ldots+\frac{\infty}{\rho}$ to the a's, we get

$$\frac{(I-a_1)(I-a_2)\ldots(I-a_{r-1})(I-a_{r+1})\ldots(I-a_n)}{(a_r-a_1)(a_r-a_2)\ldots(a_r-a_{r-1})(a_r-a_{r+1})\ldots(a_r-a_n)}$$

$$=\frac{\rho I\prod_{s=1}^{s=\infty}\left(1-\frac{\rho^2I^2}{s^2}\right)}{1-\frac{\rho I}{r}}\div \text{Lt}_{x=r}\frac{x\prod_{s=1}^{s=\infty}\left(1-\frac{x^2}{s^2}\right)}{1-\frac{x}{r}},$$

i.e.

$$\frac{\sin(\pi\rho I)}{\pi\left(1-\frac{\rho I}{r}\right)}\div\frac{\pi\cos r\pi}{-\frac{1}{r}},$$

or

$$\frac{(-1)^{r+1}}{\pi}\frac{\sin(\pi\rho I)}{1-\frac{\rho I}{r}}.$$

We have then

$$f(x)=\frac{1}{r}\sum_{r=-\infty}^{r=+\infty}\frac{(-1)^{r+1}}{r}\phi\left(\frac{r\pi x}{\rho}\right)\frac{\sin(\pi\rho I)}{1-\frac{\rho I}{r}}\frac{f}{\phi}\ldots.(B),$$

where $\dfrac{\sin(\pi\rho I)}{1-\frac{\rho I}{r}}$ is in each term of the series to be expanded in an ascending series of powers of I, and $\dfrac{f^{(r)}(0)}{\phi^{(r)}(0)}$ substituted for $I^r\dfrac{f}{\phi}$.

It is seen from the manner of formation of (B), that the expansions of the two sides in ascending powers of x must be identical. None of the coefficients $\dfrac{\sin\pi\rho I}{1-\frac{\rho I}{r}}\dfrac{f}{\phi}$ can become

infinite by reason of the proviso that $\phi(x)$ should contain all the powers of x that $f(x)$ does, so that $\dfrac{f^{(r)}(0)}{\phi^{(r)}(0)}$ cannot be infinite. The coefficient as r increases to infinity becomes

$$\sin(\pi\rho I)\frac{f}{\phi},$$

and the series at its extremities becomes coincident with the series

$$\frac{\sin\pi\rho I}{\pi}\frac{f}{\phi}\sum_{r=-\infty}^{r=+\infty}\frac{(-1)^r}{r}\phi\left(\frac{x}{\rho}\right),$$

so that the series

$$\sum_{r=-\infty}^{r=+\infty}\frac{(-1)^r}{r}\phi\left(\frac{rx}{\rho}\right)$$

must be convergent if (B) is.

If the terms $\phi\left(\frac{sx}{\rho}\right)$ and $\phi\left(-\frac{sx}{\rho}\right)$ are omitted from the otherwise continuous series, we get, denoting the defective summation by S,

$$f(x)=\frac{1}{\pi}\mathop{S}_{r=-\infty}^{r=+\infty}\frac{(-1)^{r+1}}{r}\left(1-\frac{r^2}{s^2}\right)\phi\left(\frac{rx}{\rho}\right)\frac{\sin(\pi\rho I)}{\left(1-\frac{\rho I}{r}\right)\left(1-\frac{\rho^2I^2}{r}\right)}\frac{f}{\phi}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(B'),$$

so that the convergency of the series depends on that of

$$\mathop{S}_{r=-\infty}^{r=+\infty}\frac{(-1)^r}{r}\left(1-\frac{r^2}{s^2}\right)\phi\left(\frac{rx}{\rho}\right),$$

and (B') will be convergent if the series

$$\mathop{S}_{r=-\infty}^{r=+\infty}(-1)^r\phi\left(\frac{rx}{\rho}\right).$$

is convergent.

The formula (B) may be applied to prove Fourier's series. This mode of proof is of course merely the third chapter of Fourier's *Heat* shortened by the use of symbols.

Let $$\phi(x)=\sin\pi x,$$

then we may represent any odd function in terms of a series

obtained by giving an infinite number of values to r in

$$A_r \sin \frac{r\pi x}{\rho}.$$

The coefficient A_r depends on *all* the values attached to r, and in order to find a series

$$\Sigma A_r \sin \frac{r\pi x}{\rho},$$

which when expanded in ascending powers of x, shall be identical with the expansion of a function $f(x)$, it is not necessary that the values of r should be integral and continuous from $-\infty$ to $+\infty$, as in Fourier's series. It is interesting to first give to r *all* integral values from $-\infty$ to $+\infty$ and deduce Fourier's series, and then leaving a gap in the series to apply (B') and seek the reason why this defective series cannot be used for the complete representation of a function, when the trigonometrical functions of the series are not expanded in ascending powers of x.

Since $f(x)$ is odd, only odd powers in the expansion of

$$\frac{\sin(\pi\rho I)}{1 - \frac{\rho I}{r}}$$

will have any effect when performed on

$$\frac{f(0)}{\sin(\pi 0)},$$

and $$I^{2r+1} \frac{f(0)}{\sin(\pi 0)} = (-1)^r \frac{f^{(2r+1)}(0)}{\pi^{2r+1}},$$

showing that

$$\frac{\sin(\pi\rho I)}{1 - \frac{\rho I}{r}} \frac{f(0)}{\sin(\pi 0)} = -\sqrt{(-1)} \frac{\sin\{\rho D\sqrt{(-1)}\}}{1 - \frac{\rho D\sqrt{(-1)}}{r\pi}} f(0)$$

$$= -\sqrt{(-1)}\, e^{-\frac{r\pi 0}{\rho}\sqrt{(-1)}} \times \frac{\sin\{\rho D\sqrt{(-1)} + r\pi\}}{-\frac{\rho D}{r\pi}\sqrt{(-1)}} f(0)\, e^{\frac{r\pi 0}{\rho}\sqrt{(-1)}}$$

$$= (-1)^r \frac{r\pi}{\rho} \frac{\sin\{\rho D\sqrt{(-1)}\}}{D} f(0)\, e^{\frac{r\pi 0}{\rho}\sqrt{(-1)}};$$

and since the result of performing the operation can contain only even powers of $\sqrt{(-1)}$, we may change the sign of $\sqrt{(-1)}$ in the last equation, add the two equations and take the mean. There then results

$$\frac{\sin(\pi\rho I)}{1-\frac{\rho I}{r}}\frac{f(0)}{\sin\pi 0}=(-1)^r\frac{r\pi}{\rho}\frac{\sin\{\rho D\sqrt{(-1)}\}}{D}f(0)\sin\frac{r\pi 0}{\rho}$$

$$=(-1)^{r+1}\frac{r\pi}{2\rho}\frac{\rho^{\rho D}-\rho^{-\rho D}}{D}f(0)\sin\frac{r\pi 0}{\rho}$$

$$=(-1)^{r+1}\frac{r\pi}{2\rho}\int_{-\rho}^{+\rho}f(v)\sin\frac{r\pi v}{\rho}\,dv.$$

Substituting in (1) we obtain

$$f(x)=\frac{1}{2\rho}\sum_{-\infty}^{+\infty}{}'\sin\frac{r\pi x}{\rho}\int_{-\rho}^{+\rho}f(v)\sin\frac{r\pi v}{\rho}\,dv\ldots\ldots\ldots(2),$$

the ordinary Fourier's series for $f(x)$ an odd function.

In the same way we may deduce the series for $f(x)$ an even function, and then by combining the two results we may get Fourier's general series.

It is to be noticed that (2) can only be regarded as asserting the equivalence of its two numbers developed in ascending powers of x, and numerical results obtained by substituting any value of x, the right-hand member can only be correct on account of the convergency of the series.

The successful employment of the result of substituting $\sin\pi x$ for $\phi(x)$ in (B') is seen to be impossible on account of the divergency of the series

$$\underset{r=-\infty}{\overset{r=+\infty}{S}}(-1)^r r\sin\frac{r\pi x}{\rho}.$$

The defective series however is not necessarily useless in general, for suppose $\phi(x)=e^{-x^2}$, then the series

$$\underset{r=-\infty}{\overset{r=+\infty}{S}}(-1)^r re^{-r^2x^2}$$

being convergent the defective series might be used for obtaining numerical results, although not with such rapidity of convergence as the complete series.

The series (B') becomes, when $\phi(x) = \sin \pi x$,

$$f(x) = \frac{1}{2\rho} \overset{r=+\infty}{\underset{r=-\infty}{S}} \sin \frac{r\pi x}{\rho} \left(1 - \frac{r^2}{s^2}\right)$$

$$\times \frac{\left(1 - \frac{r^2}{s^2} + \frac{\rho^2 D^2}{s^2\pi^2}\right)\int_{0-\rho}^{0+\rho} dv f(v) \sin \frac{r\pi v}{\rho} + \frac{2r\rho}{s^2\pi}\left[f(v)\cos\frac{r\pi v}{\rho}\right]_{0-\rho}^{0+\rho}}{\left(1 - \frac{r^2}{s^2} + \frac{\rho^2 D^2}{s^2\pi^2}\right)^2 + \frac{4r^2\rho^2}{s^4\pi^2} D^2},$$

which on putting $s = \infty$, reduces to (2).

We can now see why it is necessary to leave no gap in the general Fourier series.

To place the necessity in a more forcible light, suppose we had means by which we could produce the harmonic constituents of the sound given off by a vibrating string, save one constituent only. Then, however we tried to arrange the intensities of the constituents in order to make up the complete sound, no success could be attained unless the intensities of the higher harmonics be made enormously great, and in fact they would have at last to become infinite. This we could not do, so that we *must* be able to produce the wanting harmonic, that is to say, the complete representation of a note given by a string contains *all* the harmonics of the lowest note.

Lastly there remains the difficulty, that if we have

$$f(x) = \overset{r=+\infty}{\underset{r=-\infty}{S}} Ar \sin \frac{r\pi x}{\rho} \quad\text{..................}(3),$$

then if we multiply by $\sin \frac{r\pi x}{\rho}$ and integrate with respect to x between the limits $\pm\rho$, then we have

$$A_r = \frac{1}{\rho}\int_{-\rho}^{+\rho} f(x) \sin \frac{r\pi x}{\rho}.$$

But when we arrive at this result we assume that the series on the right-hand side of (3) is convergent, which has been seen to be incompatible with the gap in the series.

ON RECIPROCANTS.

By *J. J. Sylvester*.

In a note on Invariant Derivatives in the September number of the *Messenger* I have given a definition and examples of reciprocants.

If in any of the forms at the end of the postscript to the note we restore to $a, b, c, \ldots$ their values $\delta_x y, \delta^2_x y, \delta^3_x y, \ldots$ any such function divided by a certain power of $\delta_x y$ will change its sign, but otherwise remain unaltered when x and y are interchanged. The index of that power is the degree added to half the weight and will be called the index of the reciprocant. Any product of i of such reciprocants will be a reciprocant of the same kind or contrary kind to those in the table (subsequent to a) according as i is odd or even. In the latter case the interchange of x and y will leave the function absolutely unaltered. Reciprocants which cause a change of sign will be said to be of an odd, those which cause no change of sign of an even character. Any linear function of reciprocants of the same weight, degree, and *character* will be itself a reciprocant of that character, but reciprocants of opposite characters cannot be combined to form a new reciprocant: those of an odd character may be regarded as analogous to skew, those of an even character to non-skew seminvariants; the rule against combining forms of opposite characters becomes superfluous in the case of seminvariants, because those that offer themselves for combination as having the same weight and degree must of necessity be of like character. Any reciprocant being given there is a simple *ex post facto* rule for assigning its character without any knowledge of the mode of its genesis, viz. its character is odd or even according as the smallest number of letters other than a in any of its terms is odd or even. Thus the *character* of a reciprocal whose leading term is a^2e, or ab^2e, or $abce$ is odd; that of one whose *leading* term is abe or abf is even, as is also that of the remarkable reciprocal $bd - 5c^2$ in which no power of a appears.

A further important distinction between the two theories* is that there are two linear reciprocants a and b but only one linear seminvariant. As an illustration of the combinatorial *law* of *like character* it will be seen that if we operate upon

* *i.e.* of reciprocants and invariants.

$2ac - 3b^2$ with the operator

$$a(b\delta a + c\delta b) - 3b,$$

we obtain a new reciprocant

$$2ad - 10bc + 9b^3,$$

of which the character is the same as that of b^3, viz. both are odd; we may therefore add $-9b^3$ to the latter expression, and then dividing out by $2a$ there results the reciprocant $ad - 5bc$, but we cannot combine $2ac - 3b^2$ with b^2 because these two reciprocants are of opposite characters.

Again, remembering that a is of an even and b of an odd character, the three reciprocants

$$-\tfrac{45}{4}b^4, \quad 5(ac - \tfrac{3}{2}b^2)^2, \quad 3ab(ad - 5bc)$$

are all of an even order, hence we may add them together and divide the sum by a^2, which gives the new reciprocant $3bd - 5c^2$ a form not containing the first letter a.

No seminvariant exists, nor, except the one just given $bd - 5c^2$, have I been able to discover any other reciprocant in which the first letter does not make its appearance.*

The infinite progression of odd reciprocants with the leading terms

$$ac, \quad ad, \quad a.a.e, \quad a.a.f, \quad a.a.a.g, \quad a.a.a.h, \ \ldots$$

will easily be seen to exist by virtue of the general theorem that any reciprocant of degree, extent, and weight (say briefly of *dew* i, j, w) gives birth to two others of the same character as its own one of *dew* $i+1, j+1, w+2$, the other of *dew* $i+1, j+2, w+3$.

For let $\tfrac{1}{2}w + i = \lambda,$

then denoting the operator

$$b\delta_a + c\delta_b + \ldots \text{ by } \Omega,$$

and the result of the action of Ω upon itself $(\Omega *)^2$, which is in fact $\Omega^2 + \Omega_2$ (Ω_2 meaning $c\delta_a + d\delta_b + \ldots$); $(a\Omega - \lambda b)R$ will obviously be a reciprocant of *dew* $i+1, j+1, w+2$, and will give rise to a second reciprocant

$$\{a\Omega - (\lambda + \tfrac{3}{2})b\}(a\Omega - \lambda b)R,$$

which is

$$a^2(\Omega *)^2 - (2\lambda + \tfrac{1}{2})ab\Omega R - \lambda acR + (\lambda^2 + \tfrac{3}{2}\lambda)b^2R,$$

the last term of this being a reciprocant of the same character as the entire expression may be omitted, and dividing out the

* Since the above went to press I have made the capital discovery that there are an infinite number of such reciprocants, and that all those of a given weight, extent and degree may be obtained by aid of a certain quadratico-linear partial differential equation.

residue by a we obtain the second new reciprocant

$$\{a\,(\Omega^*)^2 - (2\lambda + \tfrac{1}{2})\,b - \lambda c\}\,R,$$

which will be of *dew* $i+1$, $j+2$, $w+3$, as was to be shown.

It is easy to see that every reciprocant must be a rational integral function of the forms above stated commencing with a, b, $2ac-3b^2$ (whose *dew*'s are alternately of the form i, $2i-1$, $3i-2$, i, $i-2$, $3i-1$) divided by some power of a. For if any reciprocant contains only the letters $a, b, \ldots h, k, l$, l it may be expressed as a rational integral function of the protomorph in which l first appears and of the letters $a, b, \ldots k$ divided by a power of a, and consequently the reciprocant may be so expressed, and continually repeating this process of substitution it follows that the reciprocant will be a rational integer of the protomorphs exclusively divided by a power of a*: this of course will necessarily be found only to contain combinations of like character; we already know the converse that the sum of all combinations of like character of the protomorphs is a reciprocant.† If any homogeneous reciprocant consists of portions of unlike degree (although of the same index) it is obvious that each portion must be itself a reciprocant, for if P, P', P'' ... be such portions, $P+P'+P''$... must be identical with $\Pi + \Pi' + \Pi'' + \ldots$ when Π, Π', Π'' ... are the same functions of α, β, γ ... (*i.e.* $\delta_y x$, $\delta^2_y x$, $\delta^3_y x$...) that

* The proof that every seminvariant is a rational integral function of the protomorphs is very similar: any proposed seminvariant is by the method employed in the text shown to be at worst a function of the protomorphs and of b; but the terms involving any power of b must disappear because no identical equation can connect seminvariants with a non-seminvariant b. In the text we see in like manner that any given reciprocant may be reduced to the form $H+K$, where H and K are protomorphic combinations of opposite character, so that one of them will disappear.

† Another general mode of generating a class of reciprocants would be to express any function of $a, b, c, \ldots$ say $\phi\,(a, b, c, \ldots)$ under the form $\psi\,(\alpha, \beta, \gamma, \ldots)$. The product $\phi\,(a, b, c, \ldots)\,\psi\,(a, b, c, \ldots)$, or its numerator, will then obviously be a reciprocant. To take a simple example,

$$c = \frac{d^3y}{dx^3} = -\frac{\dfrac{dx}{dy}\cdot\dfrac{d^3x}{dy^3} - 3\left(\dfrac{d^2x}{dy^2}\right)^2}{\left(\dfrac{dx}{dy}\right)^5} = -\alpha\gamma + 3\beta^2 \div \alpha^5.$$

Hence, by the rule laid down, $c\,(ac-3b^2)$, *i.e.* ac^2-3b^2c ought to be a reciprocant, which is right, for it is equal to $(2ac-3b^2)^2 - 9b^4$ divided by a multiple of a. The law that the factors of seminvariants must be seminvariants cannot be extended to the theory of reciprocants. In this case the factors may some or none of them be reciprocants, and the others on reciprocation exchange forms monocyclically or polycyclically with one another. [*I add the remark that this is not true of pure reciprocants, i.e. those in which* $\dfrac{dy}{dx}$ *does not appear. Every factor of a pure reciprocant must be itself a reciprocant.*]

P, P', P''... are of a, b, c.... If then we make

$$P - a^{2\lambda}\Pi = \Delta, \quad P' - a^{2\lambda}\Pi' = \Delta' \ldots,$$

we have $\Delta + \Delta' + \Delta'' + \ldots$ identically zero.

But P, P'... being of the same index but different degrees must be of different weights, and consequently Δ, Δ', ... are of different weights. Hence we must have $\Delta = 0$, $\Delta' = 0$, &c., as was to be shown.

It follows from this that every reciprocating function, whatever may be obtained by an algebraical combination of the protomorphs, and consequently by an algebraical combination of the forms

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\right)^i \log y,$$

and that we should gain nothing in generality by operating with successive operators of the form

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x \phi_1 \cdot\right), \quad \left(\frac{1}{y'^{\frac{1}{2}}}\delta_x \phi_2 \cdot\right), \quad \ldots$$

where ϕ_1, ϕ_2, ... are arbitrary functions of $y' \pm \frac{1}{y'}$ instead of with the simple operator $\frac{1}{y'^{\frac{1}{2}}}\delta_x$ continually repeated.

The results of using the more general operators would only amount to algebraical combinations of the results obtained from the simple forms

$$\left(\frac{1}{y'^{\frac{1}{2}}}\delta_x\right)^i \log y',$$

where i may take all values from zero to infinity.* It will be noticed that if H, K are two reciprocants of opposite character $\log y' K$ will be one of the same character.

As in the case of seminvariants so also reciprocants would in extent contain only a finite number of ground-forms; but furthermore for reciprocants limited in degree the number of ground-forms will also be finite. Whether reciprocants which are irreducible for a given extent ever cease to be so and become reducible when the order is increased, as is the case with seminvariants, remains to be seen.†

In order to facilitate the verification of the results obtained and to be obtained it may be well to express the successive derivatives of x in regard to y in terms of those

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* This is not true of homogeneous reciprocants.

† I have since found that this *is* true for reciprocants, as for seminvariants.

of y in regard to x, *i.e.* of $\alpha, \beta, \gamma, \ldots$ in terms of $a, b, c, \ldots$ as shown in the following short table.

$$\begin{array}{ll}
 & \div \\
a = \alpha & \alpha^2, \\
b = -\beta & \alpha^3, \\
c = -\alpha\gamma + 3\beta^2 & \alpha^5, \\
d = -\alpha^2\delta + 10\alpha\beta\gamma - 15\beta^3 & \alpha^7, \\
e = -\alpha^3\varepsilon + 15\alpha^2\beta\delta + 10\alpha^2\gamma^2 - 105\alpha\beta^2\gamma + 105\beta^4 & \alpha^9, \\
\left.\begin{array}{l} f = -\alpha^4\zeta + 21\alpha^3\beta\varepsilon + 35\alpha^3\gamma\delta - 210\alpha^2\beta^2\delta \\ \qquad\qquad - 280\alpha^2\beta\gamma^2 + 1260\alpha\beta^3\gamma - 945\beta^5 \end{array}\right\} & \alpha^{11}, \\
\left.\begin{array}{l} g = -\alpha^5\eta + 28\alpha^4\beta\zeta + 56\alpha^4\gamma\varepsilon + 35\alpha^4\delta^2 - 378\alpha^3\beta^2\varepsilon - 1260\alpha^3\beta\gamma\delta \\ \quad + 3150\alpha^2\beta^3\delta - 280\alpha^3\gamma^3 + 6300\alpha^2\beta^2\gamma^2 - 17325\alpha\beta^4\gamma + 10395\beta^6 \end{array}\right\} & \alpha^{13},
\end{array}$$

where $a, b, c, d, e, f, \ldots$ represent the successive derivatives of y with respect to x; and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \ldots$ of x with respect to y.

In any subsequent paper on reciprocants in this Journal, I shall make the absolutely necessary transliteration referred to in a preceding footnote, replacing the present letters $a, b, c, d, \ldots$ by the letters $t, a, b, c, \ldots$ or possibly, for reasons which carry great weight, by the expressions

$$t, \quad 2a, \quad 2.3b, \quad 2.3.4.c, \ldots$$

ON THE ZETA FUNCTION.

By *J. W. L. Glaisher.*

Forms of the Zeta function, §§ 1, 2.

§ 1. Jacobi's Zeta-function $Z(x)$, defined by the equation

$$Z(x) = -\frac{E}{K}x + \int_0^x \mathrm{dn}^2 x\,dx,$$

is periodic with respect to $2K$, but only quasi-periodic with respect to $2iK'$, viz.

$$Z(x + 2K) = \quad Z(x),$$

$$Z(x + 2iK') = -\frac{i\pi}{K} + Z(x);$$

and, in general m and n being any integers,

$$Z(x+2mK+2niK')=-\frac{ni\pi}{K}+Z(x).$$

Consider a function $\zeta(x)$ derived from $Z(x)$ by the equation

$$\zeta(x)=\frac{\pi x}{2KK'}+Z(x);$$

we find
$$\zeta(x+2K)=\frac{\pi}{K'}+\zeta(x),$$

$$\zeta(x+2iK')=\zeta(x);$$

and in general

$$\zeta(x+2mK+2niK')=\frac{m\pi}{K'}+\zeta(x).$$

This function is therefore periodic with respect to $2iK'$ and quasi-periodic with respect to $2iK'$.

Consider also a function $\mathrm{z}(x)$ derived from $Z(x)$ by the equation

$$\mathrm{z}(x)=\frac{\pi x}{4KK'}+Z(x),$$

so that $\mathrm{z}(x)$ and $\zeta(x)$ are connected by the equation

$$\mathrm{z}(x)=-\frac{\pi x}{4KK'}+\zeta(x).$$

We find
$$\mathrm{z}(x+2K)=\frac{\pi}{2K'}+\mathrm{z}(x),$$

$$\mathrm{z}(x+2iK')=-\frac{i\pi}{2K}+\mathrm{z}(x);$$

and in general

$$\mathrm{z}(x+2mK+2niK')=\frac{m\pi}{2K'}-\frac{ni\pi}{2K}+\mathrm{z}(x).$$

The function $\mathrm{z}(x)$ is therefore quasi-periodic with regard to both $2K$ and $2iK'$, and is such that corresponding to an increase of argument $2K$ the increase of the function is $\frac{\pi}{2K'}$,

and that corresponding to an increase of argument $2iK'$ the increase of the function is $-\frac{i\pi}{2K}$.*

It will be noticed that

$$\mathrm{z}(x) = \tfrac{1}{2}\{Z(x) + \zeta(x)\}.$$

§ 2. The three forms of the Zeta-function to which this paper principally relates are the functions

$$\mathrm{iz}\,x, \quad \mathrm{gz}\,x, \quad \mathrm{ez}\,x,$$

derived from $Z(x)$ by the equations

$$\mathrm{iz}\,x = \frac{I}{K}x + Z(x),$$

$$\mathrm{gz}\,x = \frac{G}{K}x + Z(x),$$

$$\mathrm{ez}\,x = \frac{E}{K}x + Z(x),$$

where $\qquad I = E - K,$

and $\qquad G = E - k'^2 K.$

Denoting by accented letters the same functions of k'^2 that the same letters unaccented are of k^2, so that

$$I' = E' - K',$$

and $\qquad G' = E' - k^2 K,$

we find

$$\mathrm{iz}\,(x + 2K) = 2I + \mathrm{iz}\,x,$$
$$\mathrm{gz}\,(x + 2K) = 2G + \mathrm{gz}\,x,$$
$$\mathrm{ez}\,(x + 2K) = 2E + \mathrm{ez}\,x;$$

$$\mathrm{iz}\,(x + 2iK') = -2iE' + \mathrm{iz}\,x,$$
$$\mathrm{gz}\,(x + 2iK') = -2iG' + \mathrm{gz}\,x,$$
$$\mathrm{ez}\,(x + 2iK') = -2iI' + \mathrm{ez}\,x;$$

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* It may be remarked that the function

$$\phi(x) = \frac{4KK'}{\pi}\,\mathrm{z}(x).$$

is such that $\qquad \phi(x + 2mK + 2niK') = 2mK - 2niK' + \phi(x).$

and in general

$$\mathrm{iz}\,(x+2mK+2niK')=2mI-2niE'+\mathrm{iz}\,x,$$

$$\mathrm{gz}\,(x+2mK+2niK')=2mG-2niG'+\mathrm{gz}\,x,$$

$$\mathrm{ez}\,(x+2mK+2niK')=2mE-2niI'+\mathrm{ez}\,x.$$

The three functions are therefore quasi-periodic with regard to both $2K$ and $2iK'$; and there exists a sort of quasi-symmetrical relation between an increment $2K$ or $2iK'$ of the argument and the corresponding increment of the function. The functions $\mathrm{iz}\,x$ and $\mathrm{ez}\,x$ form a pair of exactly similar functions in which I and E, and E' and I', are interchanged; but $\mathrm{gz}\,x$ stands by itself and is such that corresponding to an increase of argument $2K$ the increase of the function is $2G$ and corresponding to an increase of argument, $2iK'$ the increase of the function is $-2iG'$.

Increase of the argument by K, iK' or $K+iK'$, §§ 3, 4.

§ 3. We find

$$Z(x+K)=Z(x)-k^2\frac{\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x},$$

$$\mathrm{z}\,(x+K)=\frac{\pi}{4K'}+\mathrm{z}\,(x)-k^2\frac{\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x},$$

$$\zeta\,(x+K)=\frac{\pi}{2K'}+\zeta\,(x)-k^2\frac{\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x};$$

$$Z(x+iK')=-\frac{i\pi}{2K}+Z(x)-\frac{\mathrm{sn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}\,x},$$

$$\mathrm{z}\,(x+iK')=-\frac{i\pi}{4K}+\mathrm{z}\,(x)-\frac{\mathrm{sn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}\,x},$$

$$\zeta\,(x+iK')=\zeta\,(x)-\frac{\mathrm{sn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}\,x};$$

$$Z(x+K+iK')=-\frac{i\pi}{2K}+Z(x)+\frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x},$$

$$\mathrm{z}\,(x+K+iK')=\frac{\pi}{4K'}-\frac{i\pi}{4K}+\mathrm{z}\,(x)+\frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x},$$

$$\zeta\,(x+K+iK')=\frac{\pi}{2K'}+\zeta\,(x)+\frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x}.$$

§ 4. The corresponding formulæ in the case of the functions iz x, gz x, ez x are:

$$\mathrm{iz}\,(x+K) = I + \mathrm{iz}\,x - k^2\frac{\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x},$$

$$\mathrm{gz}\,(x+K) = G + \mathrm{gz}\,x - k^2\frac{\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x},$$

$$\mathrm{ez}\,(x+K) = E + \mathrm{ez}\,x - k^2\frac{\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x};$$

$$\mathrm{iz}\,(x+iK') = -iE' + \mathrm{iz}\,x - \frac{\mathrm{sn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}\,x},$$

$$\mathrm{gz}\,(x+iK') = -iG' + \mathrm{gz}\,x - \frac{\mathrm{sn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}\,x},$$

$$\mathrm{ez}\,(x+iK') = -iI' + \mathrm{ez}\,x - \frac{\mathrm{sn}\,x\,\mathrm{dn}\,x}{\mathrm{cn}\,x};$$

$$\mathrm{iz}\,(x+K+iK') = I - iE' + \mathrm{iz}\,x + \frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x},$$

$$\mathrm{gz}\,(x+K+iK') = G - iG' + \mathrm{gz}\,x + \frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x},$$

$$\mathrm{ez}\,(x+K+iK') = E - iI' + \mathrm{ez}\,x + \frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x}.$$

Functions of the argument ix, §§ 5, 6.

§ 5. The formulæ giving $Z(ix)$, $\mathrm{z}\,(ix)$ and $\zeta(ix)$ in terms of functions of the argument x are:

$$Z(ix) = -\frac{i\pi x}{2KK'} - iZ(x,k') + i\,\frac{\mathrm{sn}\,(x,k')\,\mathrm{dn}\,(x,k')}{\mathrm{cn}\,(x,k')},$$

$$\mathrm{z}\,(ix) = \qquad - i\,\mathrm{z}\,(x,k') + i\,\frac{\mathrm{sn}\,(x,k')\,\mathrm{dn}\,(x,k')}{\mathrm{cn}\,(x,k')},$$

$$\zeta\,(ix) = \frac{i\pi x}{2KK'} - i\zeta\,(x,k') + i\,\frac{\mathrm{sn}\,(x,k')\,\mathrm{dn}\,(x,k')}{\mathrm{cn}\,(x,k')}.$$

These equations may be written also in the forms:

$$Z(ix) = -\ i\zeta(x, k') + i\,\frac{\operatorname{sn}(x, k')\operatorname{dn}(x, k')}{\operatorname{cn}(x, k')},$$

$$\mathrm{z}(ix) = -\ i\mathrm{z}(x, k') + i\,\frac{\operatorname{sn}(x, k')\operatorname{dn}(x, k')}{\operatorname{cn}(x, k')},$$

$$\zeta(ix) = -\ iZ(x, k') + i\,\frac{\operatorname{sn}(x, k')\operatorname{dn}(x, k')}{\operatorname{cn}(x, k')}.$$

The corresponding relations in the case of the functions iz, gz, ez, are:

$$\mathrm{iz}\,ix = -i\,\mathrm{ez}(x, k') + i\,\frac{\operatorname{sn}(x, k')\operatorname{dn}(x, k')}{\operatorname{cn}(x, k')},$$

$$\mathrm{gz}\,ix = -i\,\mathrm{gz}(x, k') + i\,\frac{\operatorname{sn}(x, k')\operatorname{dn}(x, k')}{\operatorname{cn}(x, k')},$$

$$\mathrm{ez}\,ix = -i\ \mathrm{iz}(x, k') + i\,\frac{\operatorname{sn}(x, k')\operatorname{dn}(x, k')}{\operatorname{cn}(x, k')}.$$

The complementary character of the relation between iz$x$ and ez$x$ is apparent also in these equations, each of these two functions being transformed by the substitution of ix for x as argument into an expression involving the other, but in the case of gz the transformed expression involves the same function.

§6. From the formulæ in the last section we deduce

$$Z(ix) = -\ \ i\zeta(x + K',\ k'),$$
$$\mathrm{z}(ix) = -\ \ i\,\mathrm{z}(x + K',\ k'),$$
$$\zeta(ix) = -\ \ iZ(x + K',\ k');$$

and

$$\mathrm{iz}(ix) = -\ i\,\mathrm{ez}(x + K',\ k'),$$
$$\mathrm{gz}(ix) = -\,i\,\mathrm{gz}(x + K',\ k'),$$
$$\mathrm{ez}(ix) = -\ i\,\mathrm{iz}(x + K',\ k').$$

These formulæ are in effect included in the system in §§21, 22.

Values of the functions for arguments 0, K, iK', &c., §7.

§7. The tables on the next page give the values of $Z(x)$, $\mathrm{z}(x)$, $\zeta(x)$ and also of iz$x$, gz$x$, ez$x$ for the values 0, K, iK', &c. of x.

x	$Z(x)$	$\mathrm{z}(x)$	$\zeta(x)$
0	0	0	0
K	0	$\frac{1}{4}\frac{\pi}{K'}$	$\frac{1}{2}\frac{\pi}{K'}$
iK'	∞	∞	∞
$K+iK'$	$-\frac{1}{2}\frac{i\pi}{K}$	$\frac{1}{4}\frac{\pi}{K'}-\frac{1}{4}\frac{i\pi}{K}$	$\frac{1}{2}\frac{\pi}{K'}$
$2K$	0	$\frac{1}{2}\frac{\pi}{K'}$	$\frac{\pi}{K'}$
$2iK'$	$-\frac{i\pi}{K}$	$-\frac{1}{2}\frac{i\pi}{K}$	0
$2K+iK'$	∞	∞	∞
$K+2iK'$	$-\frac{i\pi}{K}$	$\frac{1}{4}\frac{\pi}{K'}-\frac{1}{2}\frac{i\pi}{K}$	$\frac{1}{2}\frac{\pi}{K'}$
$2K+2iK'$	$-\frac{i\pi}{K}$	$\frac{\pi}{K'}-\frac{i\pi}{K}$	$\frac{\pi}{K'}$

x	$\mathrm{iz}\,x$	$\mathrm{gz}\,x$	$\mathrm{ez}\,x$
0	0	0	0
K	I	G	E
iK'	∞	∞	∞
$K+iK'$	$I-iE'$	$G-iG'$	$E-iI'$
$2K$	$2I$	$2G$	$2E$
$2iK'$	$-2iE'$	$-2iG'$	$-2iI'$
$2K+iK'$	∞	∞	∞
$K+2iK'$	$I-2iE'$	$G-2iG'$	$E-2iI'$
$2K+2iK'$	$2I-2iE'$	$2G-2iG'$	$2E-2iI'$

The functions $Z_1(x)$, $Z_2(x)$, $Z_3(x)$, § 8.

§ 8. Consider three functions $Z_1(x)$, $Z_2(x)$, $Z_3(x)$ derived from $Z(x)$ by the equations:

$$Z_1(x) = Z(x) + \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x},$$

$$Z_2(x) = Z(x) - \frac{\operatorname{sn} x \operatorname{dn} x}{\operatorname{cn} x},$$

$$Z_3(x) = Z(x) - k^2 \frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x}.$$

It is evident therefore that we have also

$$Z_1(x) = \frac{i\pi}{2K} + Z(x + iK'),$$

$$Z_2(x) = \frac{i\pi}{2K} + Z(x + K + iK'),$$

$$Z_3(x) = Z(x + K).$$

The functions $\mathrm{z}_r(x)$ *and* $\zeta_r(x)$, § 9.

§ 9. Let $\mathrm{z}_1(x)$, $\mathrm{z}_2(x)$, $\mathrm{z}_3(x)$ and $\zeta_1(x)$, $\zeta_2(x)$, $\zeta_3(x)$ be six functions derived from $Z_1(x)$, $Z_2(x)$, $Z_3(x)$ in exactly the same manner as $\mathrm{z}(x)$ and $\zeta(x)$ were derived from $Z(x)$, viz. so that

$$\mathrm{z}_1(x) = \frac{\pi x}{4KK'} + Z_1(x),$$

$$\mathrm{z}_2(x) = \frac{\pi x}{4KK'} + Z_2(x),$$

$$\mathrm{z}_3(x) = \frac{\pi x}{4KK'} + Z_3(x);$$

and

$$\zeta_1(x) = \frac{\pi x}{2KK'} + Z_1(x),$$

$$\zeta_2(x) = \frac{\pi x}{2KK'} + Z_2(x),$$

$$\zeta_3(x) = \frac{\pi x}{2KK'} + Z_3(x).$$

Then we have

$$\mathrm{z}_1(x) = \mathrm{z}(x) + \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x},$$

$$\mathrm{z}_2(x) = \mathrm{z}(x) - \frac{\operatorname{sn} x \operatorname{dn} x}{\operatorname{cn} x},$$

$$\mathrm{z}_3(x) = \mathrm{z}(x) - k^2 \frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x};$$

and

$$\zeta_1(x) = \zeta(x) + \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x},$$

$$\zeta_2(x) = \zeta(x) - \frac{\operatorname{sn} x \operatorname{dn} x}{\operatorname{cn} x},$$

$$\zeta_3(x) = \zeta(x) - k^2 \frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x}.$$

Also

$$\mathrm{z}_1(x) = \frac{i\pi}{4K} + \mathrm{z}(x + iK'),$$

$$\mathrm{z}_2(x) = -\frac{\pi}{4K'} + \frac{i\pi}{4K} + \mathrm{z}(x + K + iK'),$$

$$\mathrm{z}_3(x) = -\frac{\pi}{4K'} + \mathrm{z}(x + K);$$

and

$$\zeta_1(x) = \zeta(x + iK'),$$

$$\zeta_2(x) = -\frac{\pi}{2K'} + \zeta(x + K + iK'),$$

$$\zeta_3(x) = -\frac{\pi}{2K'} + \zeta(x + K).$$

The functions $\mathrm{iz}_s x$, $\mathrm{gz}_s x$, $\mathrm{ez}_s x$, § 10.

§ 10. Let the functions $\mathrm{iz}_1 x$, $\mathrm{iz}_2 x$, $\mathrm{iz}_3 x$, $\mathrm{gz}_1 x$, &c. be derived from $Z_1(x)$, $Z_2(x)$, $Z_3(x)$ in the same manner as $\mathrm{iz}\, x$, $\mathrm{gz}\, x$, $\mathrm{ez}\, x$ were derived from $Z(x)$, so that

$$\mathrm{iz}_s x = \frac{I}{K} x + Z_s(x),$$

$$\mathrm{gz}_s x = \frac{G}{K} x + Z_s(x),$$

$$\mathrm{ez}_s x = \frac{E}{K} x + Z_s(x);$$

where the suffix s has the values 0, 1, 2, 3, a zero suffix being considered equivalent to the absence of a suffix, so that $\phi_0(x)$ has the same meaning as $\phi(x)$.

We then have

$$\mathrm{iz}_1 x = \mathrm{iz}\, x + \frac{\mathrm{cn}\, x\, \mathrm{dn}\, x}{\mathrm{sn}\, x},$$

$$\mathrm{iz}_2 x = \mathrm{iz}\, x - \frac{\mathrm{sn}\, x\, \mathrm{dn}\, x}{\mathrm{cn}\, x},$$

$$\mathrm{iz}_3 x = \mathrm{iz}\, x - k^2 \frac{\mathrm{sn}\, x\, \mathrm{dn}\, x}{\mathrm{dn}\, x};$$

$$\mathrm{gz}_1 x = \mathrm{gz}\, x + \frac{\mathrm{cn}\, x\, \mathrm{dn}\, x}{\mathrm{sn}\, x},$$

$$\mathrm{gz}_2 x = \mathrm{gz}\, x - \frac{\mathrm{sn}\, x\, \mathrm{dn}\, x}{\mathrm{cn}\, x},$$

$$\mathrm{gz}_3 x = \mathrm{gz}\, x - k^2 \frac{\mathrm{sn}\, x\, \mathrm{cn}\, x}{\mathrm{dn}\, x};$$

$$\mathrm{ez}_1 x = \mathrm{ez}\, x + \frac{\mathrm{cn}\, x\, \mathrm{dn}\, x}{\mathrm{sn}\, x},$$

$$\mathrm{ez}_2 x = \mathrm{ez}\, x - \frac{\mathrm{sn}\, x\, \mathrm{dn}\, x}{\mathrm{cn}\, x},$$

$$\mathrm{ez}_3 x = \mathrm{ez}\, x - k^2 \frac{\mathrm{sn}\, x\, \mathrm{cn}\, x}{\mathrm{dn}\, x};$$

and

$$\begin{aligned}
\mathrm{iz}_1 x &= iE' + \mathrm{iz}\,(x + iK'),\\
\mathrm{gz}_1 x &= iG' + \mathrm{gz}\,(x + iK'),\\
\mathrm{ez}_1 x &= iI' + \mathrm{ez}\,(x + iK');
\end{aligned}$$

$$\begin{aligned}
\mathrm{iz}_2 x &= -I + iE' + \mathrm{iz}\,(x + K + iK'),\\
\mathrm{gz}_2 x &= -G + iG' + \mathrm{gz}\,(x + K + iK'),\\
\mathrm{ez}_2 x &= -E + iI' + \mathrm{ez}\,(x + K + iK');
\end{aligned}$$

$$\begin{aligned}
\mathrm{iz}_3 x &= -I + \mathrm{iz}\,(x + K),\\
\mathrm{gz}_3 x &= -G + \mathrm{gz}\,(x + K),\\
\mathrm{ez}_3 x &= -E + \mathrm{ez}\,(x + K).
\end{aligned}$$

Correspondence between the functions, § 11.

§ 11. In the following sections, for the sake of brevity the functions $\mathrm{gz}\,x$, gz_1x, gz_2x, gz_3x will alone be considered; as the corresponding formulæ and results relating to the other functions may be at once derived from those relating to the functions $\mathrm{gz}_s x$ by replacing G, G' by I, E' in the case of the functions $\mathrm{iz}_s x$; by 0, $\frac{\pi}{2K}$ in the case of the functions $Z_s(x)$; &c., as shown below:

gz	G	G'
iz	I	E'
ez	E	I'
Z	0	$\frac{\pi}{2K}$
z	$\frac{\pi}{4K}$	$\frac{\pi}{4K'}$
ζ	$\frac{\pi}{2K'}$	0

This correspondence between the formulæ relating to the different functions is apparent in all the groups of results given in the preceding sections.

Increase of the argument by multiples of K and iK', §§ 12–14.

§ 12. The fundamental property of the function $\mathrm{gz}\,x$ is (§ 2)

$$\mathrm{gz}(x+2mK+2niK')=2mG-2niG'+\mathrm{gz}\,x,$$

and, combining this equation with the final group of results in § 10, we find

$$\begin{aligned}
\mathrm{gz}\,\{x+2mK+2niK'\}&=2mG-2niG'+\mathrm{gz}x,\\
\mathrm{gz}_1\{x+2mK+(2n+1)iK'\}&=2mG-(2n+1)iG'+\mathrm{gz}x,\\
\mathrm{gz}_2\{x+(2m+1)K+(2n+1)iK'\}&=(2m+1)G-(2n+1)iG'+\mathrm{gz}x,\\
\mathrm{gz}_3\{x+(2m+1)K+2niK'\}&=(2m+1)G-2niG'+\mathrm{gz}x.
\end{aligned}$$

Substituting $x+K$, $x+iK'$, and $x+K+iK'$ for x in § 10, we find also

$$\begin{aligned}
\mathrm{gz}\,(x+K)&=G+\mathrm{gz}_3x,\\
\mathrm{gz}_1(x+K)&=G+\mathrm{gz}_2x,\\
\mathrm{gz}_2(x+K)&=G+\mathrm{gz}_1x,\\
\mathrm{gz}_3(x+K)&=G+\mathrm{gz}\,x;
\end{aligned}$$

$$\begin{aligned}
\operatorname{gz}(x+iK') &= -iG' + \operatorname{gz}_1 x,\\
\operatorname{gz}_1(x+iK') &= -iG' + \operatorname{gz} x,\\
\operatorname{gz}_2(x+iK') &= -iG' + \operatorname{gz}_3 x,\\
\operatorname{gz}_3(x+iK') &= -iG' + \operatorname{gz}_2 x;
\end{aligned}$$

$$\begin{aligned}
\operatorname{gz}(x+K+iK') &= G - iG' + \operatorname{gz}_2 x,\\
\operatorname{gz}_1(x+K+iK') &= G - iG' + \operatorname{gz}_3 x,\\
\operatorname{gz}_2(x+K+iK') &= G - iG' + \operatorname{gz} x,\\
\operatorname{gz}_3(x+K+iK') &= G - iG' + \operatorname{gz}_1 x.
\end{aligned}$$

§ 13. By combining the last three groups of formulæ with the first we obtain the following general system of equations giving the values of the functions when the argument is increased by $mK+niK'$.

$$\begin{aligned}
\operatorname{gz}\{x+2mK+2niK'\} &= 2mG-2niG'+\operatorname{gz} x,\\
\operatorname{gz}\{x+(2m+1)K+2niK'\} &= (2m+1)G-2niG'+\operatorname{gz}_3 x,\\
\operatorname{gz}\{x+2mK+(2n+1)iK'\} &= 2mG-(2n+1)iG'+\operatorname{gz}_1 x,\\
\operatorname{gz}\{x+(2m+1)K+(2n+1)iK'\} &= (2m+1)G-(2n+1)iG'+\operatorname{gz}_2 x;
\end{aligned}$$

$$\begin{aligned}
\operatorname{gz}_1\{x+2mK+2niK'\} &= 2mG-2niG'+\operatorname{gz}_1 x,\\
\operatorname{gz}_1\{x+(2m+1)K+2niK'\} &= (2m+1)G-2niG'+\operatorname{gz}_2 x,\\
\operatorname{gz}_1\{x+2mK+(2n+1)iK'\} &= 2mG-(2n+1)iG'+\operatorname{gz} x,\\
\operatorname{gz}_1\{x+(2m+1)K+(2n+1)iK'\} &= (2m+1)G-(2n+1)iG'+\operatorname{gz}_3 x;
\end{aligned}$$

$$\begin{aligned}
\operatorname{gz}_2\{x+2mK+2niK'\} &= 2mG-2niG'+\operatorname{gz}_2 x,\\
\operatorname{gz}_2\{x+(2m+1)K+2niK'\} &= (2m+1)G-2niG'+\operatorname{gz}_1 x,\\
\operatorname{gz}_2\{x+2mK+(2n+1)iK'\} &= 2mG-(2n+1)iG'+\operatorname{gz}_3 x,\\
\operatorname{gz}_2\{x+(2m+1)K+(2n+1)iK'\} &= (2m+1)G-(2n+1)iG'+\operatorname{gz} x;
\end{aligned}$$

$$\begin{aligned}
\operatorname{gz}_3\{x+2mK+2niK'\} &= 2mG-2niG'+\operatorname{gz}_3 x,\\
\operatorname{gz}_3\{x+(2m+1)K+2niK'\} &= (2m+1)G-2niG'+\operatorname{gz} x,\\
\operatorname{gz}_3\{x+2mK+(2n+1)iK'\} &= 2mG-(2n+1)iG'+\operatorname{gz}_2 x,\\
\operatorname{gz}_3\{x+(2m+1)K+(2n+1)iK'\} &= (2m+1)G-(2n+1)iG'+\operatorname{gz}_1 x.
\end{aligned}$$

§ 14. Omitting the term independent of x, which can always be written down at sight since $mG - niG'$ corresponds to an increase of argument $mK + niK'$, these formulæ may be conveniently arranged in a table as follows:

$w = x +$	$\mathrm{gz}\,w$	$\mathrm{gz}_1 w$	$\mathrm{gz}_2 w$	$\mathrm{gz}_3 w$
$2m\ K + 2n\ iK'$	$\mathrm{gz}\,x$	$\mathrm{gz}_1 x$	$\mathrm{gz}_2 x$	$\mathrm{gz}_3 x$
$(2m+1)\,K + 2n\ iK'$	$\mathrm{gz}_3 x$	$\mathrm{gz}_2 x$	$\mathrm{gz}_1 x$	$\mathrm{gz}\,x$
$2m\ K + (2n+1)\,iK'$	$\mathrm{gz}_1 x$	$\mathrm{gz}\,x$	$\mathrm{gz}_3 x$	$\mathrm{gz}_2 x$
$(2m+1)\,K + (2n+1)\,iK'$	$\mathrm{gz}_2 x$	$\mathrm{gz}_3 x$	$\mathrm{gz}\,x$	$\mathrm{gz}_1 x$

If $\mathrm{gz}_s x$ denote any one of the four functions, s having the values 0, 1, 2, 3 and gz_0 being regarded as the same as gz (§ 10), we may include this system of equations in the single result

$$\mathrm{gz}_s (x + mK + niK') = mG - niG' + \mathrm{gz}_\sigma (x),$$

where

$$s \equiv \sigma \pmod 4), \text{ if } m \text{ and } n \text{ are both even,}$$

$$s + 2 \equiv \sigma \pmod 4) \quad ,, \quad ,, \quad \text{uneven,}$$

$$s + (-1)^s \equiv \sigma \pmod 4), \text{ if } m \text{ is even and } n \text{ is uneven,}$$

$$s - (-1)^s \equiv \sigma \pmod 4) \quad ,, \quad \text{uneven} \quad ,, \quad \text{even.}$$

Relations between the functions $\mathrm{gz}_s x$, §§ 15–19.

§ 15. The formulæ in § 12 afford the means of passing at once from a function of the arguments $x + K$, $x + iK'$, $x + K + iK'$ to a function of the argument x. As it is frequently necessary to make the reverse transition, *i.e.* to pass from a function $\mathrm{gz}_s x$ to one in which the argument is $x + K$, $x + iK'$ or $x + K + iK'$, the following list of formulæ, which is derivable at sight from § 12, is subjoined for purposes of reference.

$$\mathrm{gz}\,x = -G + \mathrm{gz}_3 (x + K),$$
$$\mathrm{gz}_1 x = -G + \mathrm{gz}_2 (x + K),$$
$$\mathrm{gz}_2 x = -G + \mathrm{gz}_1 (x + K),$$
$$\mathrm{gz}_3 x = -G + \mathrm{gz}\ (x + K);$$

$$
\begin{aligned}
\mathrm{gz}\, x &= iG' + \mathrm{gz}_1 (x + iK'),\\
\mathrm{gz}_1 x &= iG' + \mathrm{gz}\ (x + iK'),\\
\mathrm{gz}_2 x &= iG' + \mathrm{gz}_3 (x + iK'),\\
\mathrm{gz}_3 x &= iG' + \mathrm{gz}_2 (x + iK');
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{gz}\, x &= - G + iG' + \mathrm{gz}_2 (x + K + iK'),\\
\mathrm{gz}_1 x &= - G + iG' + \mathrm{gz}_3 (x + K + iK'),\\
\mathrm{gz}_2 x &= - G + iG' + \mathrm{gz}\ (x + K + iK'),\\
\mathrm{gz}_3 x &= - G + iG' + \mathrm{gz}_1 (x + K + iK').
\end{aligned}
$$

§ 16. The formulæ which enable us to pass from any function $\mathrm{gz}_s x$ to any other $\mathrm{gz}_{s'}\, x$ without change of argument are as follows:

$$
\begin{aligned}
\mathrm{gz}_1 x &= \mathrm{gz}\, x + \frac{\mathrm{cn}\, x\, \mathrm{dn}\, x}{\mathrm{sn}\, x},\\
\mathrm{gz}_2 x &= \mathrm{gz}\, x - \frac{\mathrm{sn}\, x\, \mathrm{cn}\, x}{\mathrm{cn}\, x},\\
\mathrm{gz}_3 x &= \mathrm{gz}\, x - k^2 \frac{\mathrm{sn}\, x\, \mathrm{cn}\, x}{\mathrm{dn}\, x};
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{gz}\, x &= \mathrm{gz}_1 x - \frac{\mathrm{cn}\, x\, \mathrm{dn}\, x}{\mathrm{sn}\, x},\\
\mathrm{gz}_2 x &= \mathrm{gz}_1 x - \frac{\mathrm{dn}\, x}{\mathrm{sn}\, x\, \mathrm{cn}\, x},\\
\mathrm{gz}_3 x &= \mathrm{gz}_1 x - \frac{\mathrm{cn}\, x}{\mathrm{sn}\, x\, \mathrm{dn}\, x};
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{gz}\, x &= \mathrm{gz}_2 x + \frac{\mathrm{sn}\, x\, \mathrm{dn}\, x}{\mathrm{cn}\, x},\\
\mathrm{gz}_1 x &= \mathrm{gz}_2 x + \frac{\mathrm{dn}\, x}{\mathrm{sn}\, x\, \mathrm{cn}\, x},\\
\mathrm{gz}_3 x &= \mathrm{gz}_2 x + k'^2 \frac{\mathrm{sn}\, x}{\mathrm{cn}\, x\, \mathrm{dn}\, x};
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{gz}\, x &= \mathrm{gz}_3 x + k^2 \frac{\mathrm{sn}\, x\, \mathrm{cn}\, x}{\mathrm{dn}\, x},\\
\mathrm{gz}_1 x &= \mathrm{gz}_3 x + \frac{\mathrm{cn}\, x}{\mathrm{sn}\, x\, \mathrm{dn}\, x},\\
\mathrm{gz}_2 x &= \mathrm{gz}_3 x - k'^2 \frac{\mathrm{sn}\, x}{\mathrm{cn}\, x\, \mathrm{dn}\, x}.
\end{aligned}
$$

§17. Using the functions $\mathrm{ns}\,x$, $\mathrm{dc}\,x$, $\mathrm{cd}\,x$, &c. to denote $\dfrac{1}{\mathrm{sn}\,x}$, $\dfrac{\mathrm{dn}\,x}{\mathrm{cn}\,x}$, $\dfrac{\mathrm{cn}\,x}{\mathrm{dn}\,x}$, &c.* these relations may be written in the more symmetrical form:

$$\mathrm{gz}_1 x = \mathrm{gz}\,x + \frac{\mathrm{cn}\,x\,\mathrm{dn}\,x}{\mathrm{sn}\,x} = \mathrm{gz}\,x + \frac{d}{dx}\log \mathrm{sn}\,x,$$

$$\mathrm{gz}_2 x = \mathrm{gz}\,x - \frac{\mathrm{dn}\,x\,\mathrm{sn}\,x}{\mathrm{cn}\,x} = \mathrm{gz}\,x + \frac{d}{dx}\log \mathrm{cn}\,x,$$

$$\mathrm{gz}_3 x = \mathrm{gz}\,x - k^2\frac{\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x} = \mathrm{gz}\,x + \frac{d}{dx}\log \mathrm{dn}\,x;$$

$$\mathrm{gz}\,x = \mathrm{gz}_1 x - \frac{\mathrm{cs}\,x\,\mathrm{ds}\,x}{\mathrm{ns}\,x} = \mathrm{gz}_1 x + \frac{d}{dx}\log \mathrm{ns}\,x,$$

$$\mathrm{gz}_2 x = \mathrm{gz}_1 x - \frac{\mathrm{ds}\,x\,\mathrm{ns}\,x}{\mathrm{cs}\,x} = \mathrm{gz}_1 x + \frac{d}{dx}\log \mathrm{cs}\,x,$$

$$\mathrm{gz}_3 x = \mathrm{gz}_1 x - \frac{\mathrm{ns}\,x\,\mathrm{cs}\,x}{\mathrm{ds}\,x} = \mathrm{gz}_1 x + \frac{d}{dx}\log \mathrm{ds}\,x;$$

$$\mathrm{gz}\,x = \mathrm{gz}_2 x + \frac{\mathrm{sc}\,x\,\mathrm{dc}\,x}{\mathrm{nc}\,x} = \mathrm{gz}_2 x + \frac{d}{dx}\log \mathrm{nc}\,x,$$

$$\mathrm{gz}_1 x = \mathrm{gz}_2 x + \frac{\mathrm{dc}\,x\,\mathrm{nc}\,x}{\mathrm{sc}\,x} = \mathrm{gz}_2 x + \frac{d}{dx}\log \mathrm{sc}\,x,$$

$$\mathrm{gz}_3 x = \mathrm{gz}_2 x + k'^2\frac{\mathrm{nc}\,x\,\mathrm{sc}\,x}{\mathrm{dc}\,x} = \mathrm{gz}_2 x + \frac{d}{dx}\log \mathrm{dc}\,x;$$

$$\mathrm{gz}\,x = \mathrm{gz}_3 x + k^2\frac{\mathrm{sd}\,x\,\mathrm{cd}\,x}{\mathrm{nd}\,x} = \mathrm{gz}_3 x + \frac{d}{dx}\log \mathrm{nd}\,x,$$

$$\mathrm{gz}_1 x = \mathrm{gz}_3 x + \frac{\mathrm{cd}\,x\,\mathrm{nd}\,x}{\mathrm{sd}\,x} = \mathrm{gz}_3 x + \frac{d}{dx}\log \mathrm{sd}\,x,$$

$$\mathrm{gz}_2 x = \mathrm{gz}_3 x - k'^2\frac{\mathrm{sd}\,x\,\mathrm{nd}\,x}{\mathrm{cd}\,x} = \mathrm{gz}_3 x + \frac{d}{dx}\log \mathrm{cd}\,x.$$

---

* *Messenger*, vol. xi., pp. 81–95; 120–138.

§ 18. Taking the second forms in the above equations, we may conveniently exhibit the system of relations in the following table, in which the accent denotes differentiation with respect to x:

	$\mathrm{gz}\,x +$	$\mathrm{gz}_1 x +$	$\mathrm{gz}_2 x +$	$\mathrm{gz}_3 x +$
$\mathrm{gz}\,x =$	0	$(\log \mathrm{ns}\, x)'$	$(\log \mathrm{nc}\, x)'$	$(\log \mathrm{nd}\, x)'$
$\mathrm{gz}_1 x =$	$(\log \mathrm{sn}\, x)'$	0	$(\log \mathrm{sc}\, x)'$	$(\log \mathrm{sd}\, x)'$
$\mathrm{gz}_2 x =$	$(\log \mathrm{cn}\, x)'$	$(\log \mathrm{cs}\, x)'$	0	$(\log \mathrm{cd}\, x)'$
$\mathrm{gz}_3 x =$	$(\log \mathrm{dn}\, x)'$	$(\log \mathrm{ds}\, x)'$	$(\log \mathrm{dc}\, x)'$	0

§ 19. The formulæ in the last three sections remain true when for gz is substituted any other of the functional signs iz, ez, Z, z, ζ, no further change being necessary; while in § 15 the only additional change required is the substitution of I and E', of E and I', &c., for G and G' (§ 11).

Values of the functions for arguments 0, K, iK', &c. § 20.

§ 20. The values of the functions $\mathrm{gz}\,x$, $\mathrm{gz}_1 x$, $\mathrm{gz}_2 x$, $\mathrm{gz}_3 x$ for the arguments 0, K, iK', &c. are given in the following table:

x	$\mathrm{gz}\,x$	$\mathrm{gz}_1 x$	$\mathrm{gz}_2 x$	$\mathrm{gz}_3 x$
0	0	∞	0	0
K	G	G	∞	G
iK'	∞	$-iG'$	$-iG'$	$-iG'$
$K+iK'$	$G-iG'$	$G-iG'$	$G-iG'$	∞
$2K$	$2G$	∞	$2G$	$2G$
$2iK'$	$-2iG'$	∞	$-2iG'$	$-2iG'$
$2K+iK'$	∞	$2G-iG'$	$2G-iG'$	$2G-iG'$
$K+2iK'$	$G-2iG'$	$G-2iG'$	∞	$G-2iG'$
$2K+2iK'$	$2G-2iG'$	∞	$2G-2iG'$	$2G-2iG'$

The corresponding values of the other five functions $\mathrm{iz}_s x$, $\mathrm{cz}_s x$, $Z_s(x)$, $\mathrm{z}_s(x)$, $\zeta_s(x)$ may be deduced from the values in the preceding table by replacing G, G' by I, E', by E, I', &c. as explained in § 11.

Functions of the argument ix, §§ 21, 22.

§ 21. Since, by definition,

$$Z_2(x) = Z(x) - \frac{\operatorname{sn} x \operatorname{dn} x}{\operatorname{cn} x},$$

we may express the formula:

$$Z(ix) = -\frac{i\pi x}{2KK'} - iZ(x, k') + i\frac{\operatorname{sn}(x, k')\operatorname{dn}(x, k')}{\operatorname{cn}(x, k')},$$

(§ 5) in the form

$$Z(ix) = -\frac{i\pi x}{2KK'} - iZ_2(x, k').$$

By substituting $x - iK$, $x - iK + K'$, and $x + K'$ for x in this formula, and changing the arguments to ix and to (x, k') by § 12, we obtain the formula:

$$iZ\ (ix) = \frac{\pi x}{2KK'} + Z_2(x, k'),$$

$$iZ_1(ix) = \frac{\pi x}{2KK'} + Z_1(x, k'),$$

$$iZ_2(ix) = \frac{\pi x}{2KK'} + Z\ (x, k'),$$

$$iZ_3(ix) = \frac{\pi x}{2KK'} + Z_3(x, k');$$

whence also

$$\mathrm{iz}\ (ix) = \mathrm{z}_2(x, k'),$$
$$\mathrm{iz}_1(ix) = \mathrm{z}_1(x, k'),$$
$$\mathrm{iz}_2(ix) = \mathrm{z}\ (x, k'),$$
$$\mathrm{iz}_3(ix) = \mathrm{z}_3(x, k'),$$

and

$$i\zeta\,(ix) = -\frac{\pi x}{2KK'} + \zeta_2(x, k'),$$

$$i\zeta_1(ix) = -\frac{\pi x}{2KK'} + \zeta_1(x, k'),$$

$$i\zeta_2(ix) = -\frac{\pi x}{2KK'} + \zeta\,(x, k'),$$

$$i\zeta_3(ix) = -\frac{\pi x}{2KK'} + \zeta_3(x, k').$$

Since $$\zeta_s(x) = \frac{\pi x}{2KK'} + Z_s(x),$$

the first and last of these three groups of formulæ assume the forms:

$$\begin{aligned} iZ\,(ix) &= \zeta_2(x, k'), \\ iZ_1(ix) &= \zeta_1(x, k'), \\ iZ_2(ix) &= \zeta\,(x, k'), \\ iZ_3(ix) &= \zeta_3(x, k'); \end{aligned}$$

$$\begin{aligned} i\zeta\,(ix) &= Z_2(x, k'), \\ i\zeta_1(ix) &= Z_1(x, k'), \\ i\zeta_2(ix) &= Z\,(x, k'), \\ i\zeta_3(ix) &= Z_3(x, k'). \end{aligned}$$

§ 22. The corresponding formulæ in the case of the functions $\mathrm{iz}_s x$, $\mathrm{gz}_s x$, $\mathrm{ez}_s x$ are:

$$\begin{aligned} i\ \mathrm{iz}\ ix &= \mathrm{ez}_2(x, k'), \\ i\ \mathrm{iz}_1 ix &= \mathrm{ez}_1(x, k'), \\ i\ \mathrm{iz}_2 ix &= \mathrm{ez}\ (x, k'), \\ i\ \mathrm{iz}_3 ix &= \mathrm{ez}_3(x, k'); \end{aligned}$$

$$\begin{aligned} i\ \mathrm{gz}\ ix &= \mathrm{gz}_2(x, k'), \\ i\ \mathrm{gz}_1 ix &= \mathrm{gz}_1(x, k'), \\ i\ \mathrm{gz}_2 ix &= \mathrm{gz}\ (x, k'), \\ i\ \mathrm{gz}_3 ix &= \mathrm{gz}_3(x, k'); \end{aligned}$$

$$\begin{aligned} i\ \mathrm{ez}\ ix &= \mathrm{iz}_2(x, k'), \\ i\ \mathrm{ez}_1 ix &= \mathrm{iz}_1(x, k'), \\ i\ \mathrm{ez}_2 ix &= \mathrm{iz}\ (x, k'), \\ i\ \mathrm{ez}_3 ix &= \mathrm{iz}_3(x, k'). \end{aligned}$$

The Addition-equation, §§ 23, 24.

§ 23. The well-known formula

$$Z(x)+Z(y)-Z(x+y)=k^2 \operatorname{sn} x \operatorname{sn} y \operatorname{sn}(x+y),$$

which is derivable at once from the addition-equation for the second elliptic integral, still holds good if for Z be substituted any one of the functions z, ζ, iz, gz, ez. This is evident since these functions differ from Z only by terms proportional to x. Thus, for example,

$$\operatorname{gz} x+\operatorname{gz} y-\operatorname{gz}(x+y)=k^2 \operatorname{sn} x \operatorname{sn} y \operatorname{sn}(x+y).$$

§ 24. The corresponding equations relating to the functions $\operatorname{gz}_1 x$, $\operatorname{gz}_2 x$, $\operatorname{gz}_3 x$ may be deduced by substituting in the last equation for the gz's their values in terms of gz_1's, gz_2's, and gz_3's from § 16, viz. using the formula

$$\operatorname{gz} x=\operatorname{gz}_1 x-\frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x},$$

we find, on reducing the elliptic-function expression which arises,

$$\operatorname{gz}_1 x+\operatorname{gz}_1 y-\operatorname{gz}_1(x+y)=\frac{1-\operatorname{cn} x \operatorname{cn} y \operatorname{cn}(x+y)}{\operatorname{sn} x \operatorname{sn} y \operatorname{sn}(x+y)};$$

and, similarly, we find

$$\operatorname{gz}_2 x+\operatorname{gz}_2 y-\operatorname{gz}_2(x+y)=\quad k'^2\frac{\operatorname{sn} x \operatorname{sn} y \operatorname{sn}(x+y)}{\operatorname{cn} x \operatorname{cn} y \operatorname{cn}(x+y)},$$

$$\operatorname{gz}_3 x+\operatorname{gz}_3 y-\operatorname{gz}_3(x+y)=-k^2k'^2\frac{\operatorname{sn} x \operatorname{sn} y \operatorname{sn}(x+y)}{\operatorname{dn} x \operatorname{dn} y \operatorname{dn}(x+y)}.$$

The group of four equations may be written

$$\begin{aligned}
\operatorname{gz} x+\operatorname{gz} y-\operatorname{gz}(x+y)&=\quad k^2 \operatorname{sn} x \operatorname{sn} y \operatorname{sn}(x+y),\\
\operatorname{gz}_1 x+\operatorname{gz}_1 y-\operatorname{gz}_1(x+y)&=\begin{cases}\quad\operatorname{ns} x \operatorname{ns} y \operatorname{ns}(x+y)\\ -\operatorname{cs} x \operatorname{cs} y \operatorname{cs}(x+y),\end{cases}\\
\operatorname{gz}_2 x+\operatorname{gz}_2 y-\operatorname{gz}_2(x+y)&=\quad k'^2 \operatorname{sc} x \operatorname{sc} y \operatorname{sc}(x+y),\\
\operatorname{gz}_3 x+\operatorname{gz}_3 y-\operatorname{gz}_3(x+y)&=-k^2k'^2 \operatorname{sd} x \operatorname{sd} y \operatorname{sd}(x+y).
\end{aligned}$$

In these formula any one of the other functional signs iz, ez, Z, z, ζ, may be substituted for gz.

Formulæ connecting the functions $\mathrm{iz}_{,}x$, $\mathrm{gz}_{,}x$, $\mathrm{ez}_{,}x$, § 25.

§ 25. The following table serves to express the functions $\mathrm{iz}_{,}x$, $\mathrm{gz}_{,}x$, $\mathrm{ez}_{,}x$ in terms of any one of them. Their values in terms of $Z_{,}(x)$ and $\zeta_{,}(x)$ are also added.

$\mathrm{iz}_{,}x$	$\mathrm{gz}_{,}x$	$\mathrm{ez}_{,}x$
$\frac{I}{K}x + Z_{,}(x)$	$\frac{G}{K}x + Z_{,}(x)$	$\frac{E}{K}x + Z_{,}(x)$
$\mathrm{iz}_{,}x$	$k^2x + \mathrm{iz}_{,}x$	$x + \mathrm{iz}_{,}x$
$-\ k^2x + \mathrm{gz}_{,}x$	$\mathrm{gz}_{,}x$	$k'^2x + \mathrm{gz}_{,}x$
$-\ x + \mathrm{ez}_{,}x$	$-\ k'^2x + \mathrm{ez}_{,}x$	$\mathrm{ez}_{,}x$
$-\frac{E'}{K'}x + \zeta_{,}(x)$	$-\frac{G'}{K'}x + \zeta_{,}(x)$	$-\frac{I'}{K'}x + \zeta_{,}(x)$

It may be noticed that we have also

$$
\begin{aligned}
Z_{,}(x) &= -\frac{I}{K}x + \mathrm{iz}_{,}x, \\
&= -\frac{G}{K}x + \mathrm{gz}_{,}x, \\
&= -\frac{E}{K}x + \mathrm{ez}_{,}x, \\
&= -\frac{\pi}{2KK'}x + \zeta_{,}(x);
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta_{,}(x) &= \frac{E'}{K'}x + \mathrm{iz}_{,}x, \\
&= \frac{G'}{K'}x + \mathrm{gz}_{,}x, \\
&= \frac{I'}{K'}x + \mathrm{ez}_{,}x, \\
&= \frac{\pi}{2KK'}x + Z_{,}(x).
\end{aligned}
$$

Expressions for $\mathrm{iz}_s x$, $\mathrm{gz}_s x$, $\mathrm{ez}_s x$, *as integrals*, §§ 26–28.

§ 26. The twelve functions $\mathrm{iz}_s x$, $\mathrm{gz}_s x$, $\mathrm{ez}_s x$ represent the integrals of the squares of the twelve primary elliptic functions $\mathrm{sn}\, x$, $\mathrm{ns}\, x$, $\mathrm{dc}\, x$, ...$\mathrm{nd}\, x$; viz. we have

$$\mathrm{iz}\, x = -k^2\int_0^x \mathrm{sn}^2 x\, dx,$$

$$\mathrm{gz}\, x = k^2\int_0^x \mathrm{cn}^2 x\, dx,$$

$$\mathrm{ez}\, x = \int_0^x \mathrm{dn}^2 x\, dx;$$

$$\mathrm{iz}_1 x = \frac{1}{\varepsilon} - \int_\varepsilon^x \mathrm{ns}^2 x\, dx,$$

$$\mathrm{gz}_1 x = \frac{1}{\varepsilon} - \int_\varepsilon^x \mathrm{ds}^2 x\, dx,$$

$$\mathrm{ez}_1 x = \frac{1}{\varepsilon} - \int_\varepsilon^x \mathrm{cs}^2 x\, dx;$$

$$\mathrm{iz}_2 x = -\int_0^x \mathrm{dc}^2 x\, dx,$$

$$\mathrm{gz}_2 x = -k'^2\int_0^x \mathrm{nc}^2 x\, dx,$$

$$\mathrm{ez}_2 x = -k'^2\int_0^x \mathrm{sc}^2 x\, dx;$$

$$\mathrm{iz}_3 x = -k^2\int_0^x \mathrm{cd}^2 x\, dx,$$

$$\mathrm{gz}_3 x = k^2 k'^2\int_0^x \mathrm{sd}^2 x\, dx,$$

$$\mathrm{ez}_3 x = k'^2\int_0^x \mathrm{nd}^2 x\, dx.$$

In the second group ε denotes an infinitesimal quantity.

§ 27. We may write this group in other forms in which ε does not occur, viz. we have

$$\text{iz}_1 x = \frac{1}{x} - \int_0^x \left(\text{ns}^2 x - \frac{1}{x^2}\right) dx,$$

$$\text{gz}_1 x = \frac{1}{x} - \int_0^x \left(\text{ds}^2 x - \frac{1}{x^2}\right) dx,$$

$$\text{ez}_1 x = \frac{1}{x} - \int_0^x \left(\text{cs}^2 x - \frac{1}{x^2}\right) dx,$$

and

$$\text{iz}_1 x = I + \int_x^K \text{ns}^2 x \, dx,$$

$$\text{gz}_1 x = G + \int_x^K \text{ds}^2 x \, dx,$$

$$\text{ez}_1 x = E + \int_x^K \text{cs}^2 x \, dx.$$

§ 28. The twelve functions $\text{iz}_s x$, $\text{gz}_s x$, $\text{ez}_s x$ thus form a complete group, which corresponds to the complete group of the twelve primary elliptic functions.

Although it is convenient to have a separate functional symbol for each of the twelve members of the group, it is desirable to select one of the functions as the standard form of Zeta Function, and for this purpose the function $\text{gz}\, x$ appears to be the most suitable in all respects. This function is as it were 'central' between $\text{iz}\, x$ and $\text{ez}\, x$; it is more complete in itself, and it depends upon one quantity G, instead of upon two, I and E; and these properties point it out as the natural function to be selected. It will be seen also in § 39 and the following sections that is is the function which occurs in the formulæ expressing the derivatives of the elliptic functions with respect to the modulus.

The equations which express $\text{iz}_s x$ and $\text{ez}_s x$ in terms of $\text{gz}_s x$ are:

$$\text{iz}_s x = \text{gz}_s x - k^2 x,$$

$$\text{ez}_s x = \text{gz}_s x + k'^2 x.$$

Values of $\int \mathrm{sn}^2 x\,dx$, $\int \mathrm{cn}^2 x\,dx$, &c., *in terms of* $\mathrm{gz}_r x$, §§ 29, 30.

§ 29. From § 26 we find at once

$$k^2 \int_0^x \mathrm{sn}^2 x\,dx = -\,\mathrm{gz}\,x + k^2 x,$$

$$k^2 \int_0^x \mathrm{cn}^2 x\,dx = \mathrm{gz}\,x,$$

$$\int_0^x \mathrm{dn}^2 x\,dx = \mathrm{gz}\,x + k'^2 x;$$

$$\int_0^x \left(\mathrm{ns}^2 x - \frac{1}{x^2}\right) dx = -\,\mathrm{gz}_1 x + k^2 x + \frac{1}{x},$$

$$\int_0^x \left(\mathrm{ds}^2 x - \frac{1}{x^2}\right) dx = -\,\mathrm{gz}_1 x + \frac{1}{x},$$

$$\int_0^x \left(\mathrm{cs}^2 x - \frac{1}{x^2}\right) dx = -\,\mathrm{gz}_1 x - k'^2 x + \frac{1}{x};$$

$$\int_0^x \mathrm{dc}^2 x\,dx = -\,\mathrm{gz}_2 x + k^2 x,$$

$$k'^2 \int_0^x \mathrm{nc}^2 x\,dx = -\,\mathrm{gz}_2 x,$$

$$k'^2 \int_0^x \mathrm{sc}^2 x\,dx = -\,\mathrm{gz}_2 x - k'^2 x;$$

$$k^2 \int_0^x \mathrm{cd}^2 x\,dx = -\,\mathrm{gz}_3 x + k^2 x,$$

$$k^2 k'^2 \int_0^x \mathrm{sd}^2 x\,dx = \mathrm{gz}_3 x,$$

$$k'^2 \int_0^x \mathrm{nd}^2 x\,dx = \mathrm{gz}_3 x + k'^2 x.$$

§ 30. Employing only the function gz$x$ we may write this system of results in the form:

$$k^2\int_0^x \mathrm{sn}^2 x\,dx = -\,\mathrm{gz}\,x + k^2 x,$$

$$k^2\int_0^x \mathrm{cn}^2 x\,dx = \quad \mathrm{gz}\,x,$$

$$\int_0^x \mathrm{dn}^2 x\,dx = \quad \mathrm{gz}\,x + k'^2 x;$$

$$\int_0^x \left(\mathrm{ns}^2 x - \frac{1}{x^2}\right) dx = -\,\mathrm{gz}\,x - \frac{\mathrm{cs}\,x\,\mathrm{ds}\,x}{\mathrm{ns}\,x} + k^2 x^2 + \frac{1}{x},$$

$$\int_0^x \left(\mathrm{ds}^2 x - \frac{1}{x^2}\right) dx = -\,\mathrm{gz}\,x - \frac{\mathrm{cs}\,x\,\mathrm{ds}\,x}{\mathrm{ns}\,x} + \frac{1}{x},$$

$$\int_0^x \left(\mathrm{cs}^2 x - \frac{1}{x^2}\right) dx = -\,\mathrm{gz}\,x - \frac{\mathrm{cs}\,x\,\mathrm{ds}\,x}{\mathrm{ns}\,x} - k'^2 x + \frac{1}{x},$$

$$\int_0^x \mathrm{dc}^2 x\,dx = -\,\mathrm{gz}\,x + \frac{\mathrm{sc}\,x\,\mathrm{dc}\,x}{\mathrm{nc}\,x} + k^2 x,$$

$$k'^2\int_0^x \mathrm{nc}^2 x\,dx = -\,\mathrm{gz}\,x + \frac{\mathrm{sc}\,x\,\mathrm{dc}\,x}{\mathrm{nc}\,x},$$

$$k'^2\int_0^x \mathrm{sc}^2 x\,dx = -\,\mathrm{gz}\,x + \frac{\mathrm{sc}\,x\,\mathrm{dc}\,x}{\mathrm{nc}\,x} - k'^2 x,$$

$$k^2\int_0^x \mathrm{cd}^2 x\,dx = -\,\mathrm{gz}\,x + k^2\,\frac{\mathrm{sd}\,x\,\mathrm{cd}\,x}{\mathrm{nd}\,x} + k^2 x,$$

$$k^2 k'^2\int_0^x \mathrm{sd}^2 x\,dx = \quad \mathrm{gz}\,x - k^2\,\frac{\mathrm{sd}\,x\,\mathrm{cd}\,x}{\mathrm{nd}\,x},$$

$$k'^2\int_0^x \mathrm{nd}^2 x\,dx = \quad \mathrm{gz}\,x - k^2\,\frac{\mathrm{sd}\,x\,\mathrm{cd}\,x}{\mathrm{nd}\,x} - k'^2 x.$$

This system of formulæ was given in vol. xi. p. 129*, where, however, the function $Z(x)$ was used instead of gz$x$. The great simplification produced by the use of the function gz$x$ is apparent on comparing the two modes of expressing the same system of results.

Values of $\int \mathrm{sn}^4 x\, dx$, $\int \mathrm{cn}^4 x\, dx$, &c., §§ 31–34.

§ 31. On p. 134 of vol. XI. a system of formulæ expressing $\int \mathrm{sn}^4 x\, dx$, $\int \mathrm{cn}^4 dx$, &c. in terms of $\int \mathrm{sn}^2 x\, dx$, $\int \mathrm{cn}^2 x\, dx$, &c. was given. This system of formulæ may be written:

$$\begin{aligned}
3k^4 \int \mathrm{sn}^4 x\, dx &= 2(1+k^2)\, k^2 \int \mathrm{sn}^2 x\, dx + k^2 \,\mathrm{sn}x \,\mathrm{cn}x \,\mathrm{dn}x - k^2 x, \\
3k^4 \int \mathrm{cn}^4 x\, dx &= 2(k^2-k'^2)\, k^2 \int \mathrm{cn}^2 x\, dx + k^2 \,\mathrm{sn}x \,\mathrm{cn}x \,\mathrm{dn}x + k^2 k'^2 x, \\
3 \int \mathrm{dn}^4 x\, dx &= 2(1+k'^2) \int \mathrm{dn}^2 x\, dx + k^2 \,\mathrm{sn}x \,\mathrm{cn}x \,\mathrm{dn}x - k'^2 x;
\end{aligned}$$

$$\begin{aligned}
3 \int \mathrm{ns}^4 x\, dx &= 2(1+k^2) \int \mathrm{ns}^2 x\, dx - \mathrm{ns}\,x \,\mathrm{ds}\,x \,\mathrm{cs}\,x - k^2 x, \\
3 \int \mathrm{ds}^4 x\, dx &= -2(k^2-k'^2) \int \mathrm{ds}^2 x\, dx - \mathrm{ns}\,x \,\mathrm{ds}\,x \,\mathrm{cs}\,x + k^2 k'^2 x, \\
3 \int \mathrm{cs}^4 x\, dx &= -2(1+k'^2) \int \mathrm{cs}^2 x\, dx - \mathrm{ns}\,x \,\mathrm{ds}\,x \,\mathrm{cs}\,x - k'^2 x;
\end{aligned}$$

$$\begin{aligned}
3 \int \mathrm{dc}^4 x\, dx &= 2(1+k^2) \int \mathrm{dc}^2 x\, dx + k'^2 \,\mathrm{dc}\,x \,\mathrm{nc}\,x \,\mathrm{sc}\,x - k^2 x, \\
3k'^4 \int \mathrm{nc}^4 x\, dx &= -2(k^2-k'^2)\, k'^2 \int \mathrm{nc}^2 x\, dx + k'^2 \,\mathrm{dc}\,x \,\mathrm{nc}\,x \,\mathrm{sc}\,x + k^2 k'^2 x, \\
3k'^4 \int \mathrm{sc}^4 x\, dx &= -2(1+k'^2)\, k'^2 \int \mathrm{sc}^2 x\, dx + k'^2 \,\mathrm{dc}\,x \,\mathrm{nc}\,x \,\mathrm{sc}\,x - k'^2 x;
\end{aligned}$$

$$\begin{aligned}
3k^4 \int \mathrm{cd}^4 x\, dx &= 2(1+k^2)\, k^2 \int \mathrm{cd}^2 x - k^2 k'^2 \mathrm{cd}x \,\mathrm{sd}x \,\mathrm{nd}\,x - k^2 x, \\
3k^4 k'^4 \int \mathrm{sd}^4 x\, dx &= 2(k^2-k'^2)\, k^2 k'^2 \int \mathrm{sd}^2 x - k^2 k'^2 \mathrm{cd}x \,\mathrm{sd}x \,\mathrm{nd}\,x + k^2 k'^2 x, \\
3k'^4 \int \mathrm{nd}^4 x\, dx &= 2(1+k'^2)\, k'^2 \int \mathrm{nd}^2 x - k^2 k'^2 \,\mathrm{cd}\,x \,\mathrm{sd}\,x \,\mathrm{nd}\,x - k'^2 x.
\end{aligned}$$

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* In vol. XI. only indefinite integrals were considered. If the integrals are indefinite in the second group of the system on the last page, the term $-\frac{1}{x^2}$ under the integral sign and the term $\frac{1}{x}$ on the right-hand side are of course to be omitted.

§ 32. Expressing the results by means of the functions $\text{iz}_s x$, $\text{gz}_s x$, $\text{ez}_s x$ these formulæ become:

$$3k^4\int \text{sn}^4 x\, dx = -2(1+k^2)\ \ \text{iz}\, x + k^2 \text{sn}\, x\ \text{cn}\, x\ \text{dn}\, x - k^2 x,$$

$$3k^4\int \text{cn}^4 x\, dx = 2(k^2-k'^2)\,\text{gz}\, x + k^2 \text{sn}\, x\ \text{cn}\, x\ \text{dn}\, x + k^2k'^2 x,$$

$$3\int \text{dn}^4 x\, dx = 2(1+k'^2)\ \ \text{ez}\, x + k^2 \text{sn}\, x\ \text{cn}\, x\ \text{dn}\, x - k'^2 x;$$

$$3\int \text{ns}^4 x\, dx = -2(1+k^2)\ \ \text{iz}_1 x - \text{ns}\, x\ \text{ds}\, x\ \text{cs}\, x - k^2 x,$$

$$3\int \text{ds}^4 x\, dx = 2(k^2-k'^2)\,\text{gz}_1 x - \text{ns}\, x\ \text{ds}\, x\ \text{cs}\, x + k^2k'^2 x,$$

$$3\int \text{cs}^4 x\, dx = 2(1+k'^2)\ \ \text{ez}_1 x - \text{ns}\, x\ \text{ds}\, x\ \text{cs}\, x - k'^2 x;$$

$$3\int \text{dc}^4 x\, dx = -2(1+k^2)\ \ \text{iz}_2 x + k'^2\, \text{dc}\, x\ \text{nc}\, x\ \text{sc}\, x - k^2 x,$$

$$3k'^4\int \text{nc}^4 x\, dx = 2(k^2-k'^2)\,\text{gz}_2 x + k'^2\, \text{dc}\, x\ \text{nc}\, x\ \text{sc}\, x + k^2k'^2 x,$$

$$3k'^4\int \text{sc}^4 x\, dx = 2(1+k'^2)\ \ \text{ez}_2 x + k'^2\, \text{dc}\, x\ \text{nc}\, x\ \text{sc}\, x - k'^2 x;$$

$$3k^4\int \text{cd}^4 x\, dx = -2(1+k^2)\ \ \text{iz}_3 x - k^2k'^2\, \text{cd}\, x\ \text{sd}\, x\ \text{nd}\, x - k^2 x,$$

$$3k^4k'^4\int \text{sd}^4 x\, dx = 2(k^2-k'^2)\,\text{gz}_3 x - k^2k'^2\, \text{cd}\, x\ \text{sd}\, x\ \text{nd}\, x + k^2k'^2 x,$$

$$3k'^4\int \text{nd}^4 x\, dx = 2(1+k'^2)\ \ \text{ez}_3 x - k^2k'^2\, \text{cd}\, x\ \text{sd}\, x\ \text{nd}\, x - k'^2 x.$$

§ 33. Employing only the functions $\text{gz}_s x$, we may express these formulæ as follows:

$$3k^4\int \text{sn}^4 x\, dx = -2(1+k^2)\,\text{gz}\, x + k^2 \text{sn}\, x\ \text{cn}\, x\ \text{dn}\, x + k^2(1+2k^2)\, x,$$

$$3k^4\int \text{cn}^4 x\, dx = 2(k^2-k'^2)\,\text{gz}\, x + k^2 \text{sn}\, x\ \text{cn}\, x\ \text{dn}\, x + k^2k'^2 x,$$

$$3\int \text{dn}^4 x\, dx = 2(1+k'^2)\,\text{gz}\, x + k^2 \text{sn}\, x\ \text{cn}\, x\ \text{dn}\, x + k'^2(1+2k'^2)\, x;$$

$$3\int \text{ns}^4 x\, dx = -2(1+k^2)\,\text{gz}_1 x - \text{ns}\, x\ \text{ds}\, x\ \text{cs}\, x + k^2(1+2k^2)\, x,$$

$$3\int \text{ds}^4 x\, dx = 2(k^2-k'^2)\,\text{gz}_1 x - \text{ns}\, x\ \text{ds}\, x\ \text{cs}\, x + k^2k'^2 x,$$

$$3\int \text{cs}^4 x\, dx = 2(1+k'^2)\,\text{gz}_1 x - \text{ns}\, x\ \text{ds}\, x\ \text{cs}\, x + k'^2(1+2k'^2)\, x;$$

$$3\int \mathrm{dc}^4 x\, dx = -2(1+k^2)\,\mathrm{gz}_2 x + k'^2\,\mathrm{dc}x\,\mathrm{nc}x\,\mathrm{sc}x + k^2(1+2k^2)x,$$

$$3k'^4\int \mathrm{nc}^4 x\, dx = \quad 2(k^2-k'^2)\,\mathrm{gz}_2 x + k'^2\,\mathrm{dc}\,x\,\mathrm{nc}\,x\,\mathrm{sc}\,x + k^2k'^2 x,$$

$$3k'^4\int \mathrm{sc}^4 x\, dx = \quad 2(1+k'^2)\,\mathrm{gz}_2 x + k'^2\,\mathrm{dc}x\,\mathrm{nc}x\,\mathrm{sc}x + k'^2(1+2k'^2)x\ ;$$

$$3k^4\int \mathrm{cd}^4 x\, dx = -2(1+k^2)\,\mathrm{gz}_3 x - k^2k'^2\,\mathrm{cd}x\,\mathrm{sd}x\,\mathrm{nd}x + k^2(1+2k^2)x,$$

$$3k^4k'^4\int \mathrm{sd}^4 x\, dx = \quad 2(k^2-k'^2)\,\mathrm{gz}_3 x - k^2k'^2\,\mathrm{cd}\,x\,\mathrm{sd}\,x\,\mathrm{nd}\,x + k^2k'^2 x,$$

$$3k'^4\int \mathrm{nd}^4 x\, dx = \quad 2(1+k'^2)\,\mathrm{gz}_3 x - k^2k'^2\,\mathrm{cd}x\,\mathrm{sd}x\,\mathrm{nd}x + k'^2(1+2k'^2)x.$$

§ 34. Except in the second group of each system, which relates to the functions ns, cs, ds, we may take the limits of the integrals in the last three sections to be 0 and x.

With regard to these second groups, since

$$\mathrm{ns}\,x\,\mathrm{ds}\,x\,\mathrm{cs}\,x = \frac{1}{x^3} - \frac{8(1-k^2k'^2)}{5\,!}x + \text{terms in } x^3, \text{ \&c.},$$

we have, if ε be infinitesimal,

$$3\int_\varepsilon^x \mathrm{ns}^4 x\, dx = 2(1+k^2)\int_\varepsilon^x \mathrm{ns}^2 x\, dx - \mathrm{ns}\,x\ \mathrm{ds}\,x\ \mathrm{cs}\,x - k^2 x + \frac{1}{\varepsilon^3}$$

$$= -2(1+k^2)\,\mathrm{iz}_1 x - \mathrm{ns}\,x\ \mathrm{ds}\,x\ \mathrm{cs}\,x - k^2 x + \frac{1}{\varepsilon^3} + \frac{2(1+k^2)}{\varepsilon}$$

$$= -2(1+k^2)\,\mathrm{gz}_1 x - \mathrm{ns}x\,\mathrm{ds}x\,\mathrm{cs}x + k^2(1+2k^2)x + \frac{1}{\varepsilon^3} + \frac{2(1+k^2)}{\varepsilon},$$

$$3\int_\varepsilon^x \mathrm{ds}^4 x\, dx = -2(k^2-k'^2)\int_\varepsilon^x \mathrm{ds}^2 x\, dx - \mathrm{ns}x\,\mathrm{ds}x\,\mathrm{cs}x + k^2k'^2 x + \frac{1}{\varepsilon^3}$$

$$= \quad 2(k^2-k'^2)\,\mathrm{gz}_1 x - \mathrm{ns}\,x\ \mathrm{ds}\,x\ \mathrm{cs}\,x + k^2k'^2 x + \frac{1}{\varepsilon^3} - \frac{2(k^2-k'^2)}{\varepsilon},$$

$$3\int_\varepsilon^x \mathrm{cs}^4 x = -2(1+k'^2)\int_\varepsilon^x \mathrm{cs}^2 x\, dx - \mathrm{ns}\,x\ \mathrm{ds}\,x\ \mathrm{cs}\,x - k'^2 x + \frac{1}{\varepsilon^3}$$

$$= \quad 2(1+k'^2)\,\mathrm{ez}_1 x - \mathrm{ns}\,x\ \mathrm{ds}\,x\ \mathrm{cs}\,x - k'^2 x + \frac{1}{\varepsilon^3} - \frac{2(1+k'^2)}{\varepsilon}$$

$$= \quad 2(1+k'^2)\,\mathrm{gz}_1 x - \mathrm{ns}x\,\mathrm{ds}x\,\mathrm{cs}x + k'^2(1+2k'^2)x + \frac{1}{\varepsilon^3} - \frac{2(1+k'^2)}{\varepsilon}.$$

If we take the limits of the integrals to be 0 and x, the equations assume the form

$$3\int_0^x\left\{\operatorname{ns}^4 x-\frac{1}{x^4}-\frac{2(1+k^2)}{3x^2}\right\}dx$$

$$=-2(1+k^2)\operatorname{iz}_1 x-\operatorname{ns}x\operatorname{ds}x\operatorname{cs}x-k^2x+\frac{1}{x^3}+\frac{2(1+k^2)}{x}$$

$$=-2(1+k^2)\operatorname{gz}_1 x-\operatorname{ns}x\operatorname{ds}x\operatorname{cs}x+k^2(1+2k^2)x+\frac{1}{x^3}+\frac{2(1+k^2)}{x},$$

$$3\int_0^x\left\{\operatorname{ds}^4 x-\frac{1}{x^4}+\frac{2(k^2-k'^2)}{3x^2}\right\}dx$$

$$=2(k^2-k'^2)\operatorname{gz}_1 x-\operatorname{ns}x\operatorname{ds}x\operatorname{cs}x+k^2k'^2x+\frac{1}{x^3}-\frac{2(k^2-k'^2)}{x},$$

$$3\int_0^x\left\{\operatorname{cs}^4 x-\frac{1}{x^4}+\frac{2(1+k'^2)}{3x^2}\right\}dx$$

$$=2(1+k'^2)\operatorname{ez}_1 x-\operatorname{ns}x\operatorname{ds}x\operatorname{cs}x-k'^2x+\frac{1}{x^3}-\frac{2(1+k'^2)}{x}$$

$$=2(1+k'^2)\operatorname{gz}_1 x-\operatorname{ns}x\operatorname{ds}x\operatorname{cs}x+k'^2(1+2k'^2)x+\frac{1}{x^3}-\frac{2(1+k'^2)}{x}.$$

Values of $\int\operatorname{sn}^2x\operatorname{cn}^2x\,dx$, $\int\operatorname{sn}^2x\operatorname{dn}^2x\,dx$, &c., §§ 35–37.

§ 35. By combining the formulæ for $\int\operatorname{sn}^4x\,dx$, &c. (§ 32), and those for $\int\operatorname{sn}^2x\,dx$, &c., we may obtain the following system of equations giving the values of $\int\operatorname{cn}^2x\operatorname{dn}^2x\,dx$, &c.:

$$3k^2\int\operatorname{cn}^2x\operatorname{dn}^2x\,dx=(1+k^2)\operatorname{iz}x+k^2\operatorname{sn}x\operatorname{cn}x\operatorname{dn}x+2k'^2x,$$

$$3k^2\int\operatorname{dn}^2x\operatorname{sn}^2x\,dx=(k^2-k'^2)\operatorname{gz}x-k^2\operatorname{sn}x\operatorname{cn}x\operatorname{dn}x+2k^2k'^2x,$$

$$3k^4\int\operatorname{sn}^2x\operatorname{cn}^2x\,dx=(1+k'^2)\operatorname{ez}x-k^2\operatorname{sn}x\operatorname{cn}x\operatorname{dn}x-2k'^2x;$$

$$3\int\operatorname{ds}^2x\operatorname{cs}^2x\,dx=(1+k^2)\operatorname{iz}_1x-\operatorname{ns}x\operatorname{ds}x\operatorname{cs}x+2k^2x,$$

$$3\int\operatorname{cs}^2x\operatorname{ns}^2x\,dx=-(k^2-k'^2)\operatorname{gz}_1x-\operatorname{ns}x\operatorname{ds}x\operatorname{cs}x-2k^2k'^2x,$$

$$3\int\operatorname{ns}^2x\operatorname{ds}^2x\,dx=-(1+k'^2)\operatorname{ez}_1x-\operatorname{ns}x\operatorname{ds}x\operatorname{cs}x+2k'^2x;$$

$$3k'^4 \int \mathrm{nc}^2 x\, \mathrm{sc}^2 x\, dx = \quad (1+k^2)\ \mathrm{iz}_2 x + k'^2\, \mathrm{dc}x\, \mathrm{nc}x\, \mathrm{sc}x + 2k^2 x,$$

$$3k'^2 \int \mathrm{sc}^2 x\, \mathrm{dc}^2 x\, dx = -(k^2-k'^2)\ \mathrm{gz}_2 x + k'^2\, \mathrm{dc}x\, \mathrm{nc}x\, \mathrm{sc}x - 2k^2k'^2 x,$$

$$3k'^2 \int \mathrm{dc}^2 x\, \mathrm{nc}^2 x\, dx = -(1+k'^2)\ \mathrm{ez}_2 x + k'^2\, \mathrm{dc}x\, \mathrm{nc}x\, \mathrm{sc}x + 2k'^2 x;$$

$$3k^2k'^4 \int \mathrm{sd}^2 x\, \mathrm{nd}^2 x\, dx = (1+k^2)\ \mathrm{iz}_3 x - k^2k'^2\, \mathrm{sd}\,x\, \mathrm{cd}\,x\, \mathrm{nd}\,x + 2k^2 x,$$

$$3k^2k'^2 \int \mathrm{nd}^2 x\, \mathrm{cd}^2 x\, dx = (k^2-k'^2)\ \mathrm{gz}_3 x + k^2k'^2\, \mathrm{sd}\,x\, \mathrm{cd}\,x\, \mathrm{nd}\,x + 2k^2k'^2 x,$$

$$3k^4k'^2 \int \mathrm{cd}^2 x\, \mathrm{sd}^2 x\, dx = (1+k'^2)\ \mathrm{ez}_3 x + k^2k'^2\, \mathrm{sd}\,x\, \mathrm{cd}\,x\, \mathrm{nd}\,x - 2k'^2 x.$$

§ 36. If we express the results in terms of the function gz only, we obtain the formulæ:

$$3k^2 \int \mathrm{cn}^2 x\, \mathrm{dn}^2 x\, dx = (1+k^2)\ \mathrm{gz}\,x + k^2\, \mathrm{sn}\,x\, \mathrm{cn}\,x\, \mathrm{dn}\,x + k^2k'^2 x,$$

$$3k^2 \int \mathrm{dn}^2 x\, \mathrm{sn}^2 x\, dx = (k^2-k'^2)\ \mathrm{gz}\,x - k^2\, \mathrm{sn}\,x\, \mathrm{cn}\,x\, \mathrm{dn}\,x + 2k^2k'^2 x,$$

$$3k^4 \int \mathrm{sn}^2 x\, \mathrm{cn}^2 x\, dx = (1+k'^2)\ \mathrm{gz}\,x - k^2\, \mathrm{sn}\,x\, \mathrm{cn}\,x\, \mathrm{dn}\,x - k^2k'^2 x;$$

$$3 \int \mathrm{ds}^2 x\, \mathrm{cs}^2 x\, dx = \quad (1+k^2)\ \mathrm{gz}_1 x - \mathrm{ns}\,x\, \mathrm{ds}\,x\, \mathrm{cs}\,x + k^2k'^2 x,$$

$$3 \int \mathrm{cs}^2 x\, \mathrm{ns}^2 x\, dx = -(k^2-k'^2)\ \mathrm{gz}_1 x - \mathrm{ns}\,x\, \mathrm{ds}\,x\, \mathrm{cs}\,x - 2k^2k'^2 x,$$

$$3 \int \mathrm{ns}^2 x\, \mathrm{ds}^2 x\, dx = -(1+k'^2)\ \mathrm{gz}_1 x - \mathrm{ns}\,x\, \mathrm{ds}\,x\, \mathrm{cs}\,x + k^2k'^2 x;$$

$$3k'^4 \int \mathrm{nc}^2 x\, \mathrm{sc}^2 x\, dx = \quad (1+k^2)\ \mathrm{gz}_2 x + k'^2\, \mathrm{dc}x\, \mathrm{nc}x\, \mathrm{sc}x + k^2k'^2 x,$$

$$3k'^2 \int \mathrm{sc}^2 x\, \mathrm{dc}^2 x\, dx = -(k^2-k'^2)\ \mathrm{gz}_2 x + k'^2\, \mathrm{dc}x\, \mathrm{nc}x\, \mathrm{sc}x - 2k^2k'^2 x,$$

$$3k'^2 \int \mathrm{dc}^2 x\, \mathrm{nc}^2 x\, dx = -(1+k'^2)\ \mathrm{gz}_2 x + k'^2\, \mathrm{dc}x\, \mathrm{nc}x\, \mathrm{sc}x - k^2k'^2 x;$$

$$3k^2k'^4 \int \mathrm{sd}^2 x\, \mathrm{nd}^2 x\, dx = (1+k^2)\ \mathrm{gz}_3 x - k^2k'^2\, \mathrm{cd}x\, \mathrm{sd}x\, \mathrm{nd}x + k^2k'^2 x,$$

$$3k^2k'^2 \int \mathrm{nd}^2 x\, \mathrm{cd}^2 x\, dx = (k^2-k'^2)\ \mathrm{gz}_3 x + k^2k'^2\, \mathrm{cd}x\, \mathrm{sd}x\, \mathrm{nd}x + 2k^2k'^2 x,$$

$$3k^4k'^2 \int \mathrm{cd}^2 x\, \mathrm{sd}^2 x\, dx = (1+k'^2)\ \mathrm{gz}_3 x + k^2k'^2\, \mathrm{cd}x\, \mathrm{sd}x\, \mathrm{nd}x - k^2k'^2 x.$$

§ 37. Except in the second groups, which relates to the functions ns, ds, cs, we may take the lower limit of the integrals in the last two sections to be zero, so that the limits may be supposed to be 0 and x.

If in the groups relating to ns, ds, cs we take the lower limit to be ε, where ε is infinitesimal, so that the integrals are

$$3\int_\varepsilon^x \operatorname{ds}^2 x \operatorname{cs}^2 x\, dx,$$

$$3\int_\varepsilon^x \operatorname{cs}^2 x \operatorname{ns}^2 x\, dx,$$

$$3\int_\varepsilon^x \operatorname{ns}^2 x \operatorname{ds}^2 x\, dx,$$

we must add to the right-hand members of the equations the quantities

$$\frac{1}{\varepsilon^3} - \frac{1+k^2}{\varepsilon},$$

$$\frac{1}{\varepsilon^3} + \frac{k^2-k'^2}{\varepsilon},$$

$$\frac{1}{\varepsilon^3} + \frac{1+k'^2}{\varepsilon},$$

respectively; and if we wish to take the lower limit zero, so that the integrals become

$$3\int_0^x \left\{\operatorname{ds}^2 x \operatorname{cs}^2 x - \frac{1}{x^4} + \frac{1+k^2}{3x^2}\right\} dx,$$

$$3\int_0^x \left\{\operatorname{cs}^2 x \operatorname{ns}^2 x - \frac{1}{x^4} - \frac{k^2-k'^2}{3x^2}\right\} dx,$$

$$3\int_0^x \left\{\operatorname{ns}^2 x \operatorname{ds}^2 x - \frac{1}{x^4} - \frac{1+k'^2}{3x^2}\right\} dx,$$

we must add to the right-hand members

$$\frac{1}{x^3} - \frac{1+k^2}{x},$$

$$\frac{1}{x^3} + \frac{k^2-k'^2}{x},$$

$$\frac{1}{x^3} + \frac{1+k'^2}{x},$$

respectively.

The group of functions $\operatorname{sn}^2 x$, $\operatorname{cn}^2 x$, &c., § 38.

§ 38. It will have been remarked in the preceding sections that it is not the twelve quantities $\operatorname{sn}^2 x$, $\operatorname{cn}^2 x$, &c., which so enter into the formulæ as to form groups of corresponding functions, but the twelve quantities

$$\begin{array}{ccc} -k^2 \operatorname{sn}^2 x, & k^2 \operatorname{cn}^2 x, & \operatorname{dn}^2 x, \\ \operatorname{ns}^2 x, & \operatorname{ds}^2 x, & \operatorname{cs}^2 x, \\ \operatorname{dc}^2 x, & k'^2 \operatorname{nc}^2 x, & k'^2 \operatorname{sc}^2 x, \\ -k^2 \operatorname{cd}^2 x, & k^2 k'^2 \operatorname{sd}^2 x, & k'^2 \operatorname{nd}^2 x. \end{array}$$

The fact that these quantities correspond to each other and form groups of related functions is also apparent from their expressions as q-series, and from other considerations.

Derivatives of $\operatorname{sn} x$, $\operatorname{cn} x$, &c., *with respect to the modulus*, §§ 39–43.

§ 39. The function $\operatorname{gz} x$ serves to express the derivatives $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ with respect to the modulus, viz.: the known results may be written in the form:

$$\frac{d}{dk} \operatorname{sn} x = -\frac{1}{kk'^2} \operatorname{cn} x \operatorname{dn} x \operatorname{gz} x + \frac{k}{k'^2} \operatorname{sn} x \operatorname{cn}^2 x,$$

$$\frac{d}{dk} \operatorname{cn} x = \frac{1}{kk'^2} \operatorname{sn} x \operatorname{dn} x \operatorname{gz} x - \frac{k}{k'^2} \operatorname{sn}^2 x \operatorname{cn} x,$$

$$\frac{d}{dk} \operatorname{dn} x = \frac{k}{k'^2} \operatorname{sn} x \operatorname{cn} x \operatorname{gz} x - \frac{k}{k'^2} \operatorname{sn}^2 x \operatorname{dn} x.$$

These equations may be expressed more symmetrically by taking the independent variable to be h, where h denotes k^2.

Denoting also k'^2 by h', the equations become:

$$2hh'\frac{d}{dh}\operatorname{sn}x = -\quad \operatorname{cn}x\operatorname{dn}x\operatorname{gz}x + h\operatorname{sn}x\operatorname{cn}^2x,$$

$$2hh'\frac{d}{dh}\operatorname{cn}x = \quad \operatorname{dn}x\operatorname{sn}x\operatorname{gz}x - h\operatorname{sn}^2x\operatorname{cn}x,$$

$$2hh'\frac{d}{dh}\operatorname{dn}x = h\operatorname{sn}x\operatorname{cn}x\operatorname{gz}x - h\operatorname{sn}^2x\operatorname{dn}x;$$

or, slightly changing the form of the last term,

$$2hh'\frac{d}{dh}\operatorname{sn}x = -\quad \operatorname{cn}x\operatorname{dn}x\operatorname{gz}x + h\operatorname{sn}x - h\operatorname{sn}^3x,$$

$$2hh'\frac{d}{dh}\operatorname{cn}x = \quad \operatorname{dn}x\operatorname{sn}x\operatorname{gz}x - h\operatorname{cn}x + h\operatorname{cn}^3x,$$

$$2hh'\frac{d}{dh}\operatorname{dn}x = h\operatorname{sn}x\operatorname{cn}x\operatorname{gz}x - \quad \operatorname{dn}x + \quad \operatorname{dn}^3x.$$

We may deduce at once from these formulæ by simple multiplication by $\operatorname{sn}x$, $\operatorname{cn}x$, $\operatorname{dn}x$, the corresponding equations giving the derivatives of sn^2x, &c., viz.

$$hh'\frac{d}{dh}\operatorname{sn}^2x = -\quad \operatorname{sn}x\operatorname{cn}x\operatorname{dn}x\operatorname{gz}x + h\operatorname{sn}^2x\operatorname{cn}^2x,$$

$$hh'\frac{d}{dh}\operatorname{cn}^2x = \quad \operatorname{sn}x\operatorname{cn}x\operatorname{dn}x\operatorname{gz}x - h\operatorname{sn}^2x\operatorname{cn}^2x,$$

$$hh'\frac{d}{dh}\operatorname{dn}^2x = h\operatorname{sn}x\operatorname{cn}x\operatorname{dn}x\operatorname{gz}x - h\operatorname{sn}^2x\operatorname{dn}^2x;$$

$$hh'\frac{d}{dh}\operatorname{sn}^2x = -\quad \operatorname{sn}x\operatorname{cn}x\operatorname{dn}x\operatorname{gz}x + h\operatorname{sn}^2x - h\operatorname{sn}^4x,$$

$$hh'\frac{d}{dh}\operatorname{cn}^2x = \quad \operatorname{sn}x\operatorname{cn}x\operatorname{dn}x\operatorname{gz}x - h\operatorname{cn}^2x + h\operatorname{cn}^4x,$$

$$hh'\frac{d}{dh}\operatorname{dn}^2x = h\operatorname{sn}x\operatorname{cn}x\operatorname{dn}x\operatorname{gz}x - \quad \operatorname{dn}^2x + \quad \operatorname{dn}^4x.$$

§ 40. The complete system of derivatives of the twelve functions $\operatorname{sn} x$, $\operatorname{cn} x$, &c., is given by the equations:

$$2hh' \frac{d}{dh} \operatorname{sn} x = - \operatorname{cn} x \operatorname{dn} x \operatorname{gz} x + h \operatorname{sn} x \operatorname{cn}^2 x,$$

$$2hh' \frac{d}{dh} \operatorname{cn} x = \operatorname{dn} x \operatorname{sn} x \operatorname{gz} x - h \operatorname{sn}^2 x \operatorname{cn} x,$$

$$2hh' \frac{d}{dh} \operatorname{dn} x = h \operatorname{sn} x \operatorname{cn} x \operatorname{gz} x - h \operatorname{sn}^2 x \operatorname{dn} x;$$

$$2hh' \frac{d}{dh} \operatorname{ns} x = \operatorname{ds} x \operatorname{cs} x \operatorname{gz} x - h \operatorname{cn} x \operatorname{cs} x,$$

$$2hh' \frac{d}{dh} \operatorname{ds} x = \operatorname{cs} x \operatorname{ns} x \operatorname{gz} x - h \operatorname{ds} x,$$

$$2hh' \frac{d}{dh} \operatorname{cs} x = \operatorname{ns} x \operatorname{ds} x \operatorname{gz} x - h \operatorname{cs} x;$$

$$2hh' \frac{d}{dh} \operatorname{dc} x = - h' \operatorname{nc} x \operatorname{sc} x \operatorname{gz} x,$$

$$2hh' \frac{d}{dh} \operatorname{nc} x = - \operatorname{sc} x \operatorname{dc} x \operatorname{gz} x + h \operatorname{sn} x \operatorname{sc} x,$$

$$2hh' \frac{d}{dh} \operatorname{sc} x = - \operatorname{dc} x \operatorname{nc} x \operatorname{gz} x + h \operatorname{sc} x;$$

$$2hh' \frac{d}{dh} \operatorname{cd} x = h' \operatorname{sd} x \operatorname{nd} x \operatorname{gz} x,$$

$$2hk' \frac{d}{dh} \operatorname{sd} x = - \operatorname{nd} x \operatorname{cd} x \operatorname{gz} x + h \operatorname{sd} x,$$

$$2hh' \frac{d}{dh} \operatorname{nd} x = - h \operatorname{cd} x \operatorname{sd} x \operatorname{gz} x + h \operatorname{sn} x \operatorname{sd} x.$$

§ 41. From the formulæ in the last section we also deduce immediately the system of derivatives of $\mathrm{sn}^2 x$, $\mathrm{cn}^2 x$, &c., which is as follows:

$$hh' \frac{d}{dh} \mathrm{sn}^2 x = - \quad \mathrm{sn}\, x \,\mathrm{cn}\, x \,\mathrm{dn}\, x \,\mathrm{gz}\, x + h \,\mathrm{sn}^2 x \,\mathrm{cn}^2 x,$$

$$hh' \frac{d}{dh} \mathrm{cn}^2 x = \quad \mathrm{sn}\, x \,\mathrm{cn}\, x \,\mathrm{dn}\, x \,\mathrm{gz}\, x - h \,\mathrm{sn}^2 x \,\mathrm{cn}^2 x,$$

$$hh' \frac{d}{dh} \mathrm{dn}^2 x = \quad h \,\mathrm{sn}\, x \,\mathrm{cn}\, x \,\mathrm{dn}\, x \,\mathrm{gz}\, x - h \,\mathrm{sn}^2 x \,\mathrm{dn}^2 x\,;$$

$$hh' \frac{d}{dh} \mathrm{ns}^2 x = \quad \mathrm{ns}\, x \,\mathrm{ds}\, x \,\mathrm{cs}\, x \,\mathrm{gz}\, x - h \,\mathrm{cs}^2 x,$$

$$hh' \frac{d}{dh} \mathrm{ds}^2 x = \quad \mathrm{ns}\, x \,\mathrm{ds}\, x \,\mathrm{cs}\, x \,\mathrm{gz}\, x - h \,\mathrm{ds}^2 x,$$

$$hh' \frac{d}{dh} \mathrm{cs}^2 x = \quad \mathrm{ns}\, x \,\mathrm{ds}\, x \,\mathrm{cs}\, x \,\mathrm{gz}\, x - h \,\mathrm{cs}^2 x\,;$$

$$hh' \frac{d}{dh} \mathrm{dc}^2 x = - h' \,\mathrm{dc}\, x \,\mathrm{nc}\, x \,\mathrm{sc}\, x \,\mathrm{gz}\, x,$$

$$hh' \frac{h}{dh} \mathrm{nc}^2 x = - \quad \mathrm{dc}\, x \,\mathrm{nc}\, x \,\mathrm{sc}\, x \,\mathrm{gz}\, x + h \,\mathrm{sc}^2 x,$$

$$hh' \frac{d}{dh} \mathrm{sc}^2 x = - \quad \mathrm{dc}\, x \,\mathrm{nc}\, x \,\mathrm{sc}\, x \,\mathrm{gz}\, x + h \,\mathrm{sc}^2 x\,;$$

$$hh' \frac{d}{dh} \mathrm{cd}^2 x = \quad h' \,\mathrm{cd}\, x \,\mathrm{sd}\, x \,\mathrm{nd}\, x \,\mathrm{gz}\, x,$$

$$hh' \frac{d}{dh} \mathrm{sd}^2 x = - \quad \mathrm{cd}\, x \,\mathrm{sd}\, x \,\mathrm{nd}\, x \,\mathrm{gz}\, x + h \,\mathrm{sd}^2 x,$$

$$hh' \frac{d}{dh} \mathrm{nd}^2 x = - h \quad \mathrm{cd}\, x \,\mathrm{sd}\, x \,\mathrm{nd}\, x \,\mathrm{gz}\, x + h \,\mathrm{sd}^2 x.$$

§ 42. The results contained in § 40 may be expressed also in the form:

$$2hh' \frac{d}{dh} \operatorname{sn} x = - \quad \operatorname{cn} x \operatorname{dn} x \operatorname{gz}_3 x,$$

$$2hh' \frac{d}{dh} \operatorname{cn} x = \quad \operatorname{dn} x \operatorname{sn} x \operatorname{gz}_3 x,$$

$$2hh' \frac{d}{dh} \operatorname{dn} x = \quad h \operatorname{sn} x \operatorname{cn} x \operatorname{gz}_2 x;$$

$$2hh' \frac{d}{dh} \operatorname{ns} x = \operatorname{ds} x \operatorname{cs} x \operatorname{gz}_3 x,$$

$$2hh' \frac{d}{dh} \operatorname{ds} x = \operatorname{cs} x \operatorname{ns} x \,(h' \operatorname{gz} x + h \operatorname{gz}_2 x),$$

$$2hh' \frac{d}{dh} \operatorname{cs} x = \operatorname{ns} x \operatorname{ds} x \operatorname{gz}_3 x;$$

$$2hh' \frac{d}{dh} \operatorname{dc} x = - h' \operatorname{nc} x \operatorname{sc} x \operatorname{gz} x,$$

$$2hh' \frac{d}{dh} \operatorname{nc} x = - \quad \operatorname{sc} x \operatorname{dc} x \operatorname{gz}_3 x,$$

$$2hh' \frac{d}{dh} \operatorname{sc} x = - \quad \operatorname{dc} x \operatorname{nc} x \operatorname{gz}_3 x,$$

$$2hh' \frac{d}{dh} \operatorname{cd} x = \quad h' \operatorname{sd} x \operatorname{nd} x \operatorname{gz} x,$$

$$2hh' \frac{d}{dh} \operatorname{sd} x = - \quad \operatorname{nd} x \operatorname{cd} x \,(h' \operatorname{gz} x + h \operatorname{gz}_2 x),$$

$$2hh' \frac{d}{dh} \operatorname{nd} x = - h \operatorname{cd} x \operatorname{sd} x \operatorname{gz}_2 x;$$

whence also those contained in § 41 may be expressed in the corresponding form:

$$hh' \frac{d}{dh} \operatorname{sn}^2 x = - \quad \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \operatorname{gz}_3 x,$$

$$hh' \frac{d}{dh} \operatorname{cn}^2 x = \quad \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \operatorname{gz}_3 x,$$

$$hh' \frac{d}{dh} \operatorname{dn}^2 x = \quad h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \operatorname{gz}_2 x;$$

$$hh' \frac{d}{dh} \operatorname{ns}^2 x = \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x \operatorname{gz}_3 x,$$

$$hh' \frac{d}{dh} \operatorname{ds}^2 x = \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x \,(h' \operatorname{gz} x + h \operatorname{gz}_2 x),$$

$$hh' \frac{d}{dh} \operatorname{cs}^2 x = \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x \operatorname{gz}_3 x;$$

$$hh' \frac{d}{dh} \operatorname{dc}^2 x = -h' \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x \operatorname{gz} x,$$

$$hh' \frac{d}{dh} \operatorname{nc}^2 x = - \quad \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x \operatorname{gz}_3 x,$$

$$hh' \frac{d}{dh} \operatorname{sc}^2 x = - \quad \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x \operatorname{gz}_3 x;$$

$$hh' \frac{d}{dh} \operatorname{cd}^2 x = \quad h' \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x \operatorname{gz} x,$$

$$hh' \frac{d}{dh} \operatorname{sd}^2 x = - \quad \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x \,(h' \operatorname{gz} x + h \operatorname{gz}_2 x),$$

$$hh' \frac{d}{dh} \operatorname{nd}^2 x = -h \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x \operatorname{gz}_2 x.$$

§ 43. Using d_h and d_x to denote respectively $\frac{d}{dh}$ and $\frac{d}{dx}$, we find therefore:

$$2hh' \frac{d_h \operatorname{sn} x}{d_x \operatorname{sn} x} = -\operatorname{gz}_3 x,$$

$$2hh' \frac{d_h \operatorname{cn} x}{d_x \operatorname{cn} x} = -\operatorname{gz}_3 x,$$

$$2hh' \frac{d_h \operatorname{dn} x}{d_x \operatorname{dn} x} = -\operatorname{gz}_2 x;$$

$$2hh' \frac{d_h \operatorname{ns} x}{d_x \operatorname{ns} x} = -\quad \operatorname{gz}_3 x,$$

$$2hh' \frac{d_h \operatorname{ds} x}{d_x \operatorname{ds} x} = -h' \operatorname{gz} x - h \operatorname{gz}_2 x,$$

$$2hh' \frac{d_h \operatorname{cs} x}{d_x \operatorname{cs} x} = -\quad \operatorname{gz}_3 x;$$

$$2hh' \frac{d_h \operatorname{dc} x}{d_x \operatorname{dc} x} = -\operatorname{gz} x,$$

$$2hh' \frac{d_h \operatorname{nc} x}{d_x \operatorname{nc} x} = -\operatorname{gz}_3 x,$$

$$2hh' \frac{d_h \operatorname{sc} x}{d_x \operatorname{sc} x} = -\operatorname{gz}_3 x;$$

$$2hh' \frac{d_h \operatorname{cd} x}{d_x \operatorname{cd} x} = -\quad \operatorname{gz} x,$$

$$2hh' \frac{d_h \operatorname{sd} x}{d_x \operatorname{sd} x} = -h' \operatorname{gz} x - h \operatorname{gz}_2 x,$$

$$2hh' \frac{d_h \operatorname{nd} x}{d_x \operatorname{nd} x} = -\quad \operatorname{gz}_2 x.$$

Derivatives of $k\,\mathrm{sn}\,x$, $k\,\mathrm{cn}\,x$, *&c. with respect to the modulus*, §§ 44–46.

§ 44. The derivatives with respect to h of the group of quantities $k\,\mathrm{sn}\,x$, $k\,\mathrm{cn}\,x$, $\mathrm{dn}\,x$, &c. form a more regular system of results, viz. we have:

$$2hh'\frac{d}{dh}\,k\,\mathrm{sn}\,x = -\,k\,\mathrm{cn}\,x\,\mathrm{dn}\,x\,\mathrm{gz}_2x,$$

$$2hh'\frac{d}{dh}\,k\,\mathrm{cn}\,x = k\,\mathrm{dn}\,x\,\mathrm{sn}\,x\,(k'^2\,\mathrm{gz}_1x + k^2\,\mathrm{gz}_3x),$$

$$2hh'\frac{d}{dh}\,\mathrm{dn}\,x = k^2\,\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{gz}_2x;$$

$$2hh'\frac{d}{dh}\,\mathrm{ns}\,x = \mathrm{ds}\,x\,\mathrm{cs}\,x\,\mathrm{gz}_3x,$$

$$2hh'\frac{d}{dh}\,\mathrm{ds}\,x = \mathrm{cs}\,x\,\mathrm{ns}\,x\,(k'^2\,\mathrm{gz}\,x + k^2\,\mathrm{gz}_2x),$$

$$2hh'\frac{d}{dh}\,\mathrm{cs}\,x = \mathrm{ns}\,x\,\mathrm{ds}\,x\,\mathrm{gz}_3x;$$

$$2hh'\frac{d}{dh}\,\mathrm{dc}\,x = -\,k'^2\,\mathrm{nc}\,x\,\mathrm{sc}\,x\,\mathrm{gz}\,x,$$

$$2hh'\frac{d}{dh}\,k'\,\mathrm{nc}\,x = -\,k'\,\mathrm{sc}\,x\,\mathrm{dc}\,x\,(k^2\,\mathrm{gz}_1x + k'^2\,\mathrm{gz}_3x),$$

$$2hh'\frac{d}{dh}\,k'\,\mathrm{sc}\,x = -\,k'\,\mathrm{dc}\,x\,\mathrm{nc}\,x\,\mathrm{gz}\,x;$$

$$2hh'\frac{d}{dh}\,k\,\mathrm{cd}\,x = kk'^2\,\mathrm{sd}\,x\,\mathrm{nd}\,x\,\mathrm{gz}_1x,$$

$$2hh'\frac{d}{dh}\,kk'\,\mathrm{sd}\,x = -\,kk'\,\mathrm{nd}\,x\,\mathrm{cd}\,x\,(k^2\,\mathrm{gz}\,x + k'^2\,\mathrm{gz}_2x),$$

$$2hh'\frac{d}{dh}\,k'\,\mathrm{nd}\,x = -\,k^2k'\,\mathrm{cd}\,x\,\mathrm{sd}\,x\,\mathrm{gz}_1x.$$

whence it follows that

$$2hh'\frac{d_h(k\ \mathrm{sn}\,x)}{d_x(k\ \mathrm{sn}\,x)} = -\,\mathrm{gz}_2 x,$$

$$2hh'\frac{d_h(k\ \mathrm{cn}\,x)}{d_x(k\ \mathrm{cn}\,x)} = -\,k'^2\,\mathrm{gz}_1 x - k^2\,\mathrm{gz}_3 x,$$

$$2hh'\frac{d_h(\mathrm{dn}\,x)}{d_x(\mathrm{dn}\,x)} = -\,\mathrm{gz}_2 x\,;$$

$$2hh'\frac{d_h(\mathrm{ns}\,x)}{d_x(\mathrm{ns}\,x)} = -\,\mathrm{gz}_3 x,$$

$$2hh'\frac{d_h(\mathrm{ds}\,x)}{d_x(\mathrm{ds}\,x)} = -\,k'^2\,\mathrm{gz}\,x - k^2\,\mathrm{gz}_2 x,$$

$$2hh'\frac{d_h(\mathrm{cs}\,x)}{d_x(\mathrm{cs}\,x)} = -\,\mathrm{gz}_3 x\,;$$

$$2hh'\frac{d_h(\mathrm{dc}\,x)}{d_x(\mathrm{dc}\,x)} = -\,\mathrm{gz}\,x,$$

$$2hh'\frac{d_h(k'\ \mathrm{nc}\,x)}{d_x(k'\ \mathrm{nc}\,x)} = -\,k^2\,\mathrm{gz}_1 x - k'^2\,\mathrm{gz}_3 x,$$

$$2hh'\frac{d_h(k'\ \mathrm{sc}\,x)}{d_x(k'\ \mathrm{sc}\,x)} = -\,\mathrm{gz}\,x,$$

$$2hh'\frac{d_h(k\ \mathrm{cd}\,x)}{d_x(k\ \mathrm{cd}\,x)} = -\,\mathrm{gz}_1 x,$$

$$2hh'\frac{d_h(kk'\ \mathrm{sd}\,x)}{d_x(kk'\ \mathrm{sd}\,x)} = -\,k^2\,\mathrm{gz}\,x - k'^2\,\mathrm{gz}_2 x,$$

$$2hh'\frac{d_h(k'\ \mathrm{nd}\,x)}{d_x(k'\ \mathrm{nd}\,x)} = -\,\mathrm{gz}_1 x.$$

In the above equations the letters h and h', which denote k^2 and k'^2, are used as well as k and k'. The operator $2hh'\dfrac{d}{dh}$ may be replaced by $kk'^2\dfrac{d}{dk}$, but not only are the formulæ more symmetrical when the differentiations are performed with respect to h, but in general h seems the more natural quantity to take as the independent variable, for

$\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$, ... are functions of k^2 rather than of k, *i.e.* they are rational and integral functions of k^2, and do not change sign when the sign of k is changed.

§ 45. The following equations give the derivatives of the group of quantities $k^2 \operatorname{sn}^2 x$, $k^2 \operatorname{cn}^2 x$, $\operatorname{dn}^2 x$, &c. (§ 38) with respect to h. As the first powers of k and k' do not occur, the letters h and h' are used throughout.

$$hh' \frac{d}{dh}\, h \operatorname{sn}^2 x = - h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \operatorname{gz}_2 x,$$

$$hh' \frac{d}{dh}\, h \operatorname{cn}^2 x = h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \operatorname{gz}_2 x + hh',$$

$$hh' \frac{d}{dh} \operatorname{dn}^2 x = h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \operatorname{gz}_2 x ;$$

$$hh' \frac{d}{dh} \operatorname{ns}^2 x = \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x \operatorname{gz}_3 x,$$

$$hh' \frac{d}{dh} \operatorname{ds}^2 x = \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x \operatorname{gz}_3 x - hh',$$

$$hh' \frac{d}{dh} \operatorname{cs}^2 x = \operatorname{ns} x \operatorname{ds} x \operatorname{cs} x \operatorname{gz}_3 x ;$$

$$hh' \frac{d}{dh} \operatorname{dc}^2 x = - h' \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x \operatorname{gz} x,$$

$$hh' \frac{d}{dh}\, h' \operatorname{nc}^2 x = - h' \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x \operatorname{gz} x - hh',$$

$$hh' \frac{d}{dh}\, h' \operatorname{sc}^2 x = - h' \operatorname{dc} x \operatorname{nc} x \operatorname{sc} x \operatorname{gz} x ;$$

$$hh' \frac{d}{dh}\, h \operatorname{cd}^2 x = hh' \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x \operatorname{gz}_1 x,$$

$$hh' \frac{d}{dh}\, hh' \operatorname{sd}^2 x = - hh' \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x \operatorname{gz}_1 x + hh',$$

$$hh' \frac{d}{dh}\, h' \operatorname{nd}^2 x = - hh' \operatorname{cd} x \operatorname{sd} x \operatorname{nd} x \operatorname{gz}_1 x.$$

§ 46. The middle equations of the four groups may be expressed also in the form:

$$hh'\frac{d}{dh}\,h\,\mathrm{cn}^2x = h\ \mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x\,(h'\,\mathrm{gz}_1x + h\ \mathrm{gz}_3x),$$

$$hh'\frac{d}{dh}\,\mathrm{ds}^2x = \mathrm{ns}\,x\,\mathrm{ds}\,x\ \mathrm{cs}\,x\,(h'\,\mathrm{gz}\,x + h\ \mathrm{gz}_2x),$$

$$hh'\frac{d}{dh}\,h'\,\mathrm{nc}^2x = -\ h'\ \mathrm{dc}\,x\,\mathrm{nc}\,x\ \mathrm{sc}\,x\,(h\ \mathrm{gz}_1x + h'\,\mathrm{gz}_3x),$$

$$hh'\frac{d}{dh}\,hh'\ \mathrm{sd}^2x = -\ hh'\ \mathrm{cd}\,x\ \mathrm{sd}\,x\ \mathrm{nd}\,x\,(h\ \mathrm{gz}\,x + h'\,\mathrm{gz}_2x).$$

These results, which are derivable at once from the corresponding equations in § 44, may be readily identified with the results given the last section by means of the following equations, which may be easily deduced from § 16:

$$k^2\,\mathrm{gz}\,x + k'^2\,\mathrm{gz}_2x = \mathrm{gz}_1x - \mathrm{dn}\,x\ \mathrm{ds}\,x\ \mathrm{dc}\,x,$$

$$k^2\,\mathrm{gz}_1x + k'^2\,\mathrm{gz}_3x = \mathrm{gz}\,x + k^2\,\mathrm{cn}\,x\ \mathrm{cs}\,x\ \mathrm{cd}\,x,$$

$$k^2\,\mathrm{gz}_3x + k'^2\,\mathrm{gz}_1x = \mathrm{gz}_2x + k'^2\,\mathrm{ns}\,x\ \mathrm{nc}\,x\ \mathrm{nd}\,x,$$

$$k^2\,\mathrm{gz}_2x + k'^2\,\mathrm{gz}\,x = \mathrm{gz}_3x - k^2k'^2\,\mathrm{sn}\,x\ \mathrm{sc}\,x\ \mathrm{sd}\,x.$$

It may be remarked that these equations are evidently still true if for the functions gz_s we substitute throughout any of the other functions iz_s, ez_s, Z_s, ζ_s, or z_s.

Differentiation with respect to the modulus when the argument is $\dfrac{2Kx}{\pi}$, §§ 47–50.

§ 47. In the preceding sections which relate to differentiation with respect to h, the argument x has been supposed to be independent of x, viz. for example, $\mathrm{sn}\,x$ has been regarded as a function $\mathrm{sn}\,(x, h)$ of x and h, in which these two variables are independent. In the formulæ of Elliptic Functions the argument is frequently not x but $\dfrac{2Kx}{\pi}$, and it is therefore desirable to consider the derivatives with respect to h of the functions of this form of argument.

Let u denote $\frac{2Kx}{\pi}$, x being a quantity independent of h; we then have

$$\frac{du}{dh} = \frac{2x}{\pi}\frac{dK}{dh},$$

whence, since $$\frac{dK}{dh} = \frac{G}{2hh'},$$

$$\frac{du}{dh} = \frac{Gx}{\pi hh'} = \frac{Gu}{2Khh'}.$$

Now

$$\frac{d}{dh}\operatorname{sn}(u, h) = \left[\frac{d}{du}\operatorname{sn}(u, h)\right]\frac{du}{dh} + \left[\frac{d}{dh}\operatorname{sn}(u, h)\right]$$

where the brackets are used to indicate that in performing the differentiations u and h are to be regarded as independent.

Thus

$$2hh'\frac{d}{dh}\operatorname{sn}(u, h) = \operatorname{cn} u \operatorname{dn} u \frac{Gu}{K} - \operatorname{cn} u \operatorname{dn} u \operatorname{gz}_3 u.$$

$$= -\operatorname{cn} u \operatorname{dn} u\, Z_3(u).$$

In this manner we find that the derivatives of $\operatorname{sn} u$, $\operatorname{cn} u$, &c., differ from those of $\operatorname{sn} x$, $\operatorname{cn} x$, &c. given in § 42 only by the substitution of u for x and of the functions $Z_s(x)$ for $\operatorname{gz}_s x$.

The system of formulæ is therefore as follows:

$$\left[u = \frac{2Kx}{\pi}\right]$$

$$2hh'\frac{d}{dh}\operatorname{sn} u = -\operatorname{cn} u \operatorname{dn} u\, Z_3(u),$$

$$2hh'\frac{d}{dh}\operatorname{cn} u = \operatorname{dn} u \operatorname{sn} u\, Z_3(u),$$

$$2hh'\frac{d}{dh}\operatorname{dn} u = h \operatorname{sn} u \operatorname{cn} u\, Z_2(u);$$

$$2hh'\frac{d}{dh}\operatorname{ns}u = \operatorname{ds}u\operatorname{cs}u\,Z_3(u),$$

$$2hh'\frac{d}{dh}\operatorname{ds}u = \operatorname{cs}u\operatorname{ns}u\,\{h'Z(u)+hZ_2(u)\},$$

$$2hh'\frac{d}{dh}\operatorname{cs}u = \operatorname{ns}u\operatorname{ds}u\,Z_3(u);$$

$$2hh'\frac{d}{dh}\operatorname{dc}u = -h'\operatorname{nc}u\operatorname{sc}u\,Z(u),$$

$$2hh'\frac{d}{dh}\operatorname{nc}u = -\operatorname{sc}u\operatorname{dc}u\,Z_3(u),$$

$$2hh'\frac{d}{dh}\operatorname{sc}u = -\operatorname{dc}u\operatorname{nc}u\,Z_3(u);$$

$$2hh'\frac{d}{dh}\operatorname{cd}u = h'\operatorname{sd}u\operatorname{nd}u\,Z(u),$$

$$2hh'\frac{d}{dh}\operatorname{sd}u = -\operatorname{nd}u\operatorname{sd}u\,\{h'Z(u)+hZ_2(u)\},$$

$$2hh'\frac{d}{dh}\operatorname{nd}u = -h\operatorname{sd}u\operatorname{cd}u\,Z_2(u).$$

§ 48. Corresponding to the formulæ giving the values of

$$hh'\frac{d}{dh}\operatorname{sn}^2 x,\qquad hh'\frac{d}{dh}\operatorname{cn}^2 x,\quad \&c.$$

(§ 42) and to the equations

$$2hh'\frac{d_h\operatorname{sn}x}{d_x\operatorname{sn}x} = -\operatorname{gz}_3 x,\quad \&c.$$

(§ 43), we have exactly similar results in which the argument is u, the only difference being the substitution of Zs for gzs.

The same is true also of the formulæ in §§ 44–46, and it is therefore sufficient to give, in the next two sections, only the more important systems.

§ 49. The formulæ corresponding to those in § 44 are:

$$\left[u = \frac{2Kx}{\pi}\right]$$

$$2hh' \frac{d}{dh} k \operatorname{sn} u = -k \operatorname{cn} u \operatorname{dn} u \, Z_2(u),$$

$$2hh' \frac{d}{dh} k \operatorname{cn} u = k \operatorname{dn} u \operatorname{sn} u \, \{k'^2 Z_1(u) + k^2 Z_3(u)\},$$

$$2hh' \frac{d}{dh} \operatorname{dn} u = k^2 \operatorname{sn} u \operatorname{cn} u \, Z_2(u);$$

$$2hh' \frac{d}{dh} \operatorname{ns} u = \operatorname{ds} u \operatorname{cs} u \, Z_3(u),$$

$$2hh' \frac{d}{dh} \operatorname{ds} u = \operatorname{cs} u \operatorname{ns} u \, \{k'^2 Z(u) + k^2 Z_2(u)\},$$

$$2hh' \frac{d}{dh} \operatorname{cs} u = \operatorname{ns} u \operatorname{ds} u \, Z_3(u);$$

$$2hh' \frac{d}{dh} \operatorname{dc} u = -k'^2 \operatorname{nc} u \operatorname{sc} u \, Z(u),$$

$$2hh' \frac{d}{dh} k' \operatorname{nc} u = -k' \operatorname{sc} u \operatorname{dc} u \, \{k^2 Z_1(u) + k'^2 Z_3(u)\},$$

$$2hh' \frac{d}{dh} k' \operatorname{sc} u = -k' \operatorname{dc} u \operatorname{nc} u \, Z(u);$$

$$2hh' \frac{d}{dh} k \operatorname{cd} u = kk'^2 \operatorname{sd} u \operatorname{nd} u \, Z_1(u),$$

$$2hh' \frac{d}{dh} kk' \operatorname{sd} u = -kk' \operatorname{nd} u \operatorname{cd} u \, \{k^2 Z(u) + k'^2 Z_2(u)\},$$

$$2hh' \frac{d}{dh} k' \operatorname{nd} u = -k^2 k' \operatorname{cd} u \operatorname{sd} u \, Z_1(u).$$

§ 50. The formulæ corresponding to those in § 45 are:

$$\left[u = \frac{2Kx}{\pi}\right]$$

$$\begin{aligned}
hh' \frac{d}{dh}\, h \operatorname{sn}^2 u &= - h \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u\, Z_2(u),\\
hh' \frac{d}{dh}\, h \operatorname{cn}^2 u &= h \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u\, Z_2(u) + hh',\\
hh' \frac{d}{dh} \operatorname{dn}^2 u &= h \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u\, Z_2(u);
\end{aligned}$$

$$\begin{aligned}
hh' \frac{d}{dh} \operatorname{ns}^2 u &= \operatorname{ns} u \operatorname{ds} u \operatorname{cs} u\, Z_3(u),\\
hh' \frac{d}{dh} \operatorname{ds}^2 u &= \operatorname{ns} u \operatorname{ds} u \operatorname{cs} u\, Z_3(u) - hh',\\
hh' \frac{d}{dh} \operatorname{cs}^2 u &= \operatorname{ns} u \operatorname{ds} u \operatorname{cs} u\, Z_3(u);
\end{aligned}$$

$$\begin{aligned}
hh' \frac{d}{dh} \operatorname{dc}^2 u &= - h' \operatorname{dc} u \operatorname{nc} u \operatorname{sc} u\, Z(u),\\
hh' \frac{d}{dh}\, h' \operatorname{nc}^2 u &= - h' \operatorname{dc} u \operatorname{nc} u \operatorname{sc} u\, Z(u) - hh',\\
hh' \frac{d}{dh}\, h' \operatorname{sc}^2 u &= - h' \operatorname{dc} u \operatorname{nc} u \operatorname{sc} u\, Z(u);
\end{aligned}$$

$$\begin{aligned}
hh' \frac{d}{dh}\, h \operatorname{cd}^2 u &= hh' \operatorname{cd} u \operatorname{sd} u \operatorname{nd} u\, Z_1(u),\\
hh' \frac{d}{dh}\, hh' \operatorname{sd}^2 u &= - hh' \operatorname{cd} u \operatorname{sd} u \operatorname{nd} u\, Z_1(u) + hh',\\
hh' \frac{h}{dh}\, h' \operatorname{nd}^2 u &= - hh' \operatorname{cd} u \operatorname{sd} u \operatorname{nd} u\, Z_1(u).
\end{aligned}$$

Derivatives of the functions gz,x *with respect to* h, §§ 51, 52.

§ 51. By integrating the equation

$$hh' \frac{d}{dh} \operatorname{dn}^2 x = h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \operatorname{gz} x - h \operatorname{sn}^2 x \operatorname{dn}^2 x,$$

(§ 39) with respect to x, we find

$$hh' \frac{d}{dh} \int_0^x \operatorname{dn}^2 x dx = -\tfrac{1}{2} h \operatorname{cn}^2 x \operatorname{gz} x + \tfrac{1}{2} h^2 \int_0^x \operatorname{cn}^4 x dx - h \int_0^x \operatorname{sn}^2 x \operatorname{dn}^2 x dx,$$

the first two terms on the right-hand side being obtained by integration by parts.

Substituting for $h^2 \int \operatorname{cn}^4 x\, dx$ and $h \int \operatorname{sn}^2 x \operatorname{dn}^2 x dx$ their values from §§ 33 and 36, we find

$$\begin{aligned} 2hh' \frac{d}{dh} \operatorname{iz} x = & - h \operatorname{cn}^2 x \operatorname{gz} x \\ & + \tfrac{2}{3}(h-h') \operatorname{gz} x + \tfrac{1}{3} h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x + \tfrac{1}{3} hh'x \\ & - \tfrac{2}{3}(h-h') \operatorname{gz} x + \tfrac{2}{3} h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x - \tfrac{4}{3} hh'x \\ = & - h \operatorname{cn}^2 x \operatorname{gz} x + h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x - hh'x, \end{aligned}$$

whence, since $\operatorname{iz} x = \operatorname{gz} x - hx,$

we obtain the formula

$$2hh' \frac{d}{dh} \operatorname{gz} x = - h \operatorname{cn}^2 x \operatorname{gz} x + h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x + hh'x.$$

It can be shown by § 40 that

$$2hh' \frac{d}{dh} \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x} = \left\{ \frac{1}{\operatorname{sn}^2 x} - h \operatorname{sn}^2 x \right\} \operatorname{gz} x - h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x - h \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x},$$

$$2hh' \frac{d}{dh} \frac{\operatorname{sn} x \operatorname{dn} x}{\operatorname{cn} x} = - \left\{ \frac{h'}{\operatorname{cn}^2 x} + h \operatorname{cn}^2 x \right\} \operatorname{gz} x + h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x,$$

$$2hh' \frac{d}{dh} h \frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x} = \left\{ \frac{h'}{\operatorname{dn}^2 x} - \operatorname{dn}^2 x \right\} \operatorname{gz} x + h \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x + hh' \frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x},$$

and by means of these results we may deduce from the formula giving the derivative of gz x the corresponding formulæ relating

to $\mathrm{gz}_1 x$, $\mathrm{gz}_2 x$ and $\mathrm{gz}_3 x$, the group of four equations being thus found to be:

$$2hh' \frac{d}{dh} \mathrm{gz}\, x = - h \,\mathrm{cn}^2 x \,\mathrm{gz}\, x + h \,\mathrm{sn}\, x \,\mathrm{cn}\, x \,\mathrm{dn}\, x + hh'x,$$

$$2hh' \frac{d}{dh} \mathrm{gz}_1 x = \mathrm{ds}^2 x \,\mathrm{gz}\, x - h \frac{\mathrm{cn}\, x \,\mathrm{dn}\, x}{\mathrm{sn}\, x} + hh'x,$$

$$2hh' \frac{d}{dh} \mathrm{gz}_2 x = h' \,\mathrm{nc}^2 x \,\mathrm{gz}\, x + hh'x,$$

$$2hh' \frac{d}{dh} \mathrm{gz}_3 x = - hh' \,\mathrm{sd}^2 x \,\mathrm{gz}\, x - hh' \frac{\mathrm{sn}\, x \,\mathrm{cn}\, x}{\mathrm{dn}\, x} + hh'x,$$

§ 52. In the preceding equations the derivatives of the four functions $\mathrm{gz}_r x$ are expressed in terms of $\mathrm{gz}\, x$. The results, however, admit of being exhibited in a more elegant and symmetrical form as follows:

$$2hh' \frac{d}{dh} \mathrm{gz}\, x = - h \,\mathrm{cn}^2 x \,\mathrm{gz}_2 x + hh'x,$$

$$2hh' \frac{d}{dh} \mathrm{gz}_1 x = \mathrm{ds}^2 x \,\mathrm{gz}_3 x + hh'x,$$

$$2hh' \frac{d}{dh} \mathrm{gz}_2 x = h' \,\mathrm{nc}^2 x \,\mathrm{gz}\, x + hh'x,$$

$$2hh' \frac{d}{dh} \mathrm{gz}_3 x = - hh' \,\mathrm{sd}^2 x \,\mathrm{gz}_1 x + hh'x.$$

Using an accent to denote differentiation with respect to x, we may write these results in the form:

$$2hh' \frac{d}{dh} \mathrm{gz}\, x = - \mathrm{gz}'\, x \,\mathrm{gz}_2 x + hh'x,$$

$$2hh' \frac{d}{dh} \mathrm{gz}_1 x = - \mathrm{gz}_1' x \,\mathrm{gz}_3 x + hh'x,$$

$$2hh' \frac{d}{dh} \mathrm{gz}_2 x = - \mathrm{gz}_2' x \,\mathrm{gz}\, x + hh'x,$$

$$2hh' \frac{d}{dh} \mathrm{gz}_3 x = - \mathrm{gz}_3' x \,\mathrm{gz}_1 x + hh'x;$$

from which we deduce also the group:

$$2hh'\frac{d}{dh}\,\mathrm{gz}\,x = -\,\mathrm{gz}'\,x\,\mathrm{gz}\,x + h\ \mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x + hh'x,$$

$$2hh'\frac{d}{dh}\,\mathrm{gz}_1 x = -\,\mathrm{gz}_1'x\,\mathrm{gz}_1 x - \mathrm{ns}\,x\,\mathrm{ds}\,x\,\mathrm{cs}\,x + hh'x,$$

$$2hh'\frac{d}{dh}\,\mathrm{gz}_2 x = -\,\mathrm{gz}_2'x\,\mathrm{gz}_2 x + h'\,\mathrm{dc}\,x\,\mathrm{nc}\,x\,\mathrm{sc}\,x + hh'x,$$

$$2hh'\frac{d}{dh}\,\mathrm{gz}_3 x = -\,\mathrm{gz}_3'x\,\mathrm{gz}_3 x - hh'\,\mathrm{cd}\,x\,\mathrm{sd}\,x\,\mathrm{nd}\,x + hh'x.$$

Derivatives of $\mathrm{iz}_s x$, &c., § 53.

§ 53. The derivatives of $\mathrm{iz}_s x$, $\mathrm{ez}_s x$, $Z_s(x)$, $\zeta_s(x)$ are readily deducible from those of $\mathrm{gz}_s x$; for

$$\mathrm{iz}_s x = \mathrm{gz}_s x - hx,$$

$$\mathrm{ez}_s x = \mathrm{gz}_s x + h'x,$$

$$Z_s(x) = \mathrm{gz}_s x - \frac{G}{K}x,$$

$$\zeta_s(x) = \mathrm{gz}_s x + \frac{G'}{K'}x;$$

and therefore

$$2hh'\frac{d}{dh}\,\mathrm{iz}_s x = 2hh'\frac{d}{dh}\,\mathrm{ez}_s x = 2hh'\frac{d}{dh}\,\mathrm{gz}_s x - 2hh'x,$$

$$2hh'\frac{d}{dh}\,Z_s(x) = 2hh'\frac{d}{dh}\,\mathrm{gz}_s x - hh'x + \frac{G^2}{K^2}x,$$

$$2hh'\frac{d}{dh}\,\zeta_s(x) = 2hh'\frac{d}{dh}\,\mathrm{gz}_s x - hh'x + \frac{G'^2}{K'^2}x.$$

Thus, if the functions differentiated be $\mathrm{iz}_s x$ or $\mathrm{ez}_s x$ instead of $\mathrm{gz}_s x$, the only change required in the right-hand members of the groups of formulæ given in the two preceding sections is the substitution of $-hh'x$ for $hh'x$; and if the functions be $Z_s(x)$ or $\zeta_s(x)$, the only change required is the substitution of $\dfrac{G^2}{K^2}x$ or $\dfrac{G'^2}{K'^2}x$ for $hh'x$.

For example,

$$2hh'\frac{d}{dh}Z(x) = -\ h\,\mathrm{cn}^2x\,\mathrm{gz}_2x + \frac{G^2}{K^2}x$$

$$= -\ \mathrm{gz}'x\,\mathrm{gz}_2x + \frac{G^2}{K^2}x$$

$$= -\ \mathrm{gz}'x\,\mathrm{gz}\,x + h\,\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x + \frac{G^2}{K^2}x.$$

Since $\mathrm{z}_s(x) = \frac{1}{2}\{Z_s(x) + \zeta_s(x)\}$, it is evident that when the function differentiated is $\mathrm{z}_s(x)$, the term $hh'x$ is to be replaced by $\frac{1}{2}\left\{\frac{G^2}{K^2} + \frac{G'^2}{K'^2}\right\}x$.

Values of the derivatives when the argument is $\frac{2Kx}{\pi}$, §§ 54–56.

§ 54. Denoting, as before, $\frac{2Kx}{\pi}$ by u we find, by the method of § 47,

$$2hh'\frac{d}{dh}\mathrm{gz}\,u = -\ h\,\mathrm{cn}^2u\,\mathrm{gz}_2u + hh'u + h\,\mathrm{cn}^2u\,.\,2hh'\frac{du}{dh}$$

$$= -\ h\,\mathrm{cn}^2u\,\mathrm{gz}_2u + hh'u + h\,\mathrm{cn}^2u\,\frac{Gu}{K},$$

whence $$2hh'\frac{d}{dh}\mathrm{gz}\,u = -\ h\,\mathrm{cn}^2u\,Z_2(u) + hh'u.$$

We thus obtain the group of formulæ:

$$\left[u = \frac{2Kx}{\pi}\right]$$

$$2hh'\frac{d}{dh}\mathrm{gz}\,u = -\ h\,\mathrm{cn}^2u\,Z_2(u) + hh'u,$$

$$2hh'\frac{d}{dh}\mathrm{gz}_1u = \mathrm{ds}^2u\,Z_3(u) + hh'u,$$

$$2hh'\frac{d}{dh}\mathrm{gz}_2u = h'\,\mathrm{nc}^2u\,Z(u) + hh'u,$$

$$2hh'\frac{d}{dh}\mathrm{gz}_3u = -\ hh'\,\mathrm{sd}^2u\,Z_1(u) + hh'u.$$

This group may be also written in the forms:

$$2hh'\frac{d}{dh}\,\mathrm{gz}\,u = -\,\mathrm{gz}'\,u\,Z_2(u) + hh'u,$$

$$2hh'\frac{d}{dh}\,\mathrm{gz}_1 x = -\,\mathrm{gz}_1'u\,Z_3(u) + hh'u,$$

$$2hh'\frac{d}{dh}\,\mathrm{gz}_2 u = -\,\mathrm{gz}_2'u\,Z\,(u) + hh'u,$$

$$2hh'\frac{d}{dh}\,\mathrm{gz}_3 u = -\,\mathrm{gz}_3'u\,Z_1(u) + hh'u;$$

and

$$2hh'\frac{d}{dh}\,\mathrm{gz}\,u = -\,\mathrm{gz}'\,u\,Z\,(u) + \;h\;\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u + hh'u,$$

$$2hh'\frac{d}{dh}\,\mathrm{gz}_1 u = -\,\mathrm{gz}_1'u\,Z_1(u) - \qquad \mathrm{ns}\,u\,\mathrm{ds}\,u\,\mathrm{cs}\,u + hh'u,$$

$$2hh'\frac{d}{dh}\,\mathrm{gz}_2 u = -\,\mathrm{gz}_2'u\,Z_2(u) + \;h'\;\mathrm{dc}\,u\,\mathrm{nc}\,u\,\mathrm{sc}\,u + hh'u,$$

$$2hh'\frac{d}{dh}\,\mathrm{gz}_3 u = -\,\mathrm{gz}_3'u\,Z_3(u) - hh'\,\mathrm{cd}\,u\,\mathrm{sd}\,u\,\mathrm{nd}\,u + hh'u.$$

§ 55. It will be found that if the functions differentiated be $\mathrm{iz}_s u$ or $\mathrm{ez}_s u$ instead of $\mathrm{gz}_s u$, the term $hh'u$ is to be replaced in all the formulæ by $-\dfrac{hEu}{K}$ or by $\dfrac{h'Iu}{K}$ respectively; and that if the functions be $Z_s(u)$ or $\zeta_s(u)$, the term $hh'u$ is to be replaced by zero or by $\dfrac{\pi G'u}{2KK'^2}$ respectively.

§ 56. On account of their importance and simplicity, the formulæ giving the derivatives of $Z_s(u)$ are written at full length below in all three forms:

$$2hh'\frac{d}{dh}\,Z\,(u) = -\;h\;\mathrm{cn}^2u\,Z_2(u),$$

$$2hh'\frac{d}{dh}\,Z_1(u) = \qquad \mathrm{ds}^2u\,Z_3(u),$$

$$2hh'\frac{d}{dh}\,Z_2(u) = \;h'\;\mathrm{nc}^2u\,Z\,(u),$$

$$2hh'\frac{d}{dh}\,Z_3(u) = -\,hh'\,\mathrm{sd}^2u\,Z_1(u);$$

$$2hh'\frac{d}{dh}Z(u) = -\operatorname{gz}'u\,Z_2(u),$$

$$2hh'\frac{d}{dh}Z_1(u) = -\operatorname{gz}_1'u\,Z_3(u),$$

$$2hh'\frac{d}{dh}Z_2(u) = -\operatorname{gz}_2'u\,Z(u),$$

$$2hh'\frac{d}{dh}Z_3(u) = -\operatorname{gz}_3'u\,Z_1(u);$$

$$2hh'\frac{d}{dh}Z(u) = -\operatorname{gz}'u\,Z(u) + h\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u,$$

$$2hh'\frac{d}{dh}Z_1(u) = -\operatorname{gz}_1'u\,Z_1(u) - \operatorname{ns}u\operatorname{ds}u\operatorname{cs}u,$$

$$2hh'\frac{d}{dh}Z_2(u) = -\operatorname{gz}_2'u\,Z_2(u) + h'\operatorname{dc}u\operatorname{nc}u\operatorname{sc}u,$$

$$2hh'\frac{d}{dh}Z_3(u) = -\operatorname{gz}_3'u\,Z_3(u) - hh'\operatorname{cd}u\operatorname{sd}u\operatorname{nd}u.$$

The derivatives of $(\log \operatorname{sn}x)'$, $(\log \operatorname{cn}x)'$, &c., §§ 57, 58.

§ 57. With reference to the formulæ giving the derivatives of $\frac{\operatorname{cn}x\operatorname{dn}x}{\operatorname{sn}x}$, $\frac{\operatorname{sn}x\operatorname{dn}x}{\operatorname{cn}x}$, $h\frac{\operatorname{sn}x\operatorname{cn}x}{\operatorname{dn}x}$, in § 51 it may be remarked that the results in § 52 show that

$$2hh'\frac{d}{dh}\frac{\operatorname{cn}x\operatorname{dn}x}{\operatorname{sn}x} = h\operatorname{cn}^2x\operatorname{gz}_2x + \operatorname{ds}^2x\operatorname{gz}_3x,$$

$$2hh'\frac{d}{dh}\frac{\operatorname{sn}x\operatorname{dn}x}{\operatorname{cn}x} = -h\operatorname{cn}^2x\operatorname{gz}_2x - h'\operatorname{nc}^2x\operatorname{gz}x,$$

$$2hh'\frac{d}{dh}h\frac{\operatorname{sn}x\operatorname{cn}x}{\operatorname{dn}x} = -h\operatorname{cn}^2x\operatorname{gz}_2x + hh'\operatorname{sd}^2x\operatorname{gz}_1x;$$

or,

$$2hh'\frac{d}{dh}\ \frac{\operatorname{cn}x\operatorname{dn}x}{\operatorname{sn}x} = \operatorname{gz}'x\operatorname{gz}_2x - \operatorname{gz}'_1x\operatorname{gz}_3x,$$

$$2hh'\frac{d}{dh}\ \frac{\operatorname{sn}x\operatorname{dn}x}{\operatorname{cn}x} = \operatorname{gz}_2'x\operatorname{gz}x - \operatorname{gz}'x\operatorname{gz}_2x,$$

$$2hh'\frac{d}{dh}\ h\frac{\operatorname{sn}x\operatorname{cn}x}{\operatorname{dn}x} = \operatorname{gz}_3'x\operatorname{gz}_1x - \operatorname{gz}'x\operatorname{gz}_2x.$$

§ 58. It is perhaps worth while to give the complete system of formulæ corresponding to this last group. This system may be obtained at once by means of the table in § 18, which shows that

$$\operatorname{gz}x - \operatorname{gz}_1x = (\log\operatorname{ns}x)',\quad \operatorname{gz}x - \operatorname{gz}_2x = (\log\operatorname{nc}x)',\ \text{\&c.},$$

the accent denoting differentiation with respect to x. We thus find:

$$2hh'\frac{d}{dh}(\log\operatorname{sn}x)' = \operatorname{gz}'x\operatorname{gz}_2x - \operatorname{gz}_1'x\operatorname{gz}_3x,$$

$$2hh'\frac{d}{dh}(\log\operatorname{cn}x)' = \operatorname{gz}'x\operatorname{gz}_2x - \operatorname{gz}_2'x\operatorname{gz}x,$$

$$2hh'\frac{d}{dh}(\log\operatorname{dn}x)' = \operatorname{gz}'x\operatorname{gz}_2x - \operatorname{gz}_3'x\operatorname{gz}_1x;$$

$$2hh'\frac{d}{dh}(\log\operatorname{ns}x)' = \operatorname{gz}_1'x\operatorname{gz}_3x - \operatorname{gz}'x\operatorname{gz}_2x,$$

$$2hh'\frac{d}{dh}(\log\operatorname{ds}x)' = \operatorname{gz}_1'x\operatorname{gz}_3x - \operatorname{gz}_3'x\operatorname{gz}_1x,$$

$$2hh'\frac{d}{dh}(\log\operatorname{cs}x)' = \operatorname{gz}_1'x\operatorname{gz}_3x - \operatorname{gz}_2'x\operatorname{gz}x;$$

$$2hh'\frac{d}{dh}(\log\operatorname{dc}x)' = \operatorname{gz}_2'x\operatorname{gz}x - \operatorname{gz}_3'x\operatorname{gz}_1x,$$

$$2hh'\frac{d}{dh}(\log\operatorname{nc}x)' = \operatorname{gz}_2'x\operatorname{gz}x - \operatorname{gz}'x\operatorname{gz}_2x,$$

$$2hh'\frac{d}{dh}(\log\operatorname{sc}x)' = \operatorname{gz}_2'x\operatorname{gz}x - \operatorname{gz}_1'x\operatorname{gz}_3x;$$

$$2hh'\frac{d}{dh}(\log \mathrm{cd}\, x)' = \mathrm{gz}_3'x\, \mathrm{gz}_1 x - \mathrm{gz}_2'x\, \mathrm{gz}\, x,$$

$$2hh'\frac{d}{dh}(\log \mathrm{sd}\, x)' = \mathrm{gz}_3'x\, \mathrm{gz}_1 x - \mathrm{gz}_1'x\, \mathrm{gz}_3 x,$$

$$2hh'\frac{d}{dh}(\log \mathrm{nd}\, x)' = \mathrm{gz}_3'x\, \mathrm{gz}_1 x - \mathrm{gz}'\, x\, \mathrm{gz}_2 x.$$

Expansion of the functions $\mathrm{gz}_s x$ *in ascending powers of* x.
§ 59.

§ 59. The first four terms of the expansions of the functions $\mathrm{gz}_s x$ I have found to be as follows.

$$\mathrm{gz}\, x = k^2 \left\{x - 2\frac{x^3}{3\,!} + 2^3(1+k^2)\frac{x^5}{5\,!} - 2^4(2+13k^2+2k^4)\frac{x^7}{7\,!} + \&c.\right\},$$

$$\mathrm{gz}_1 x = \frac{1}{x} + \tfrac{1}{3}(k^2 - k'^2)\,x - \tfrac{2}{15}(1-k^2k'^2)\frac{x^3}{3\,!}$$
$$+ \tfrac{8}{63}(1+k^2)(1+k'^2)(k^2-k'^2)\frac{x^5}{5\,!} - \&c.,$$

$$\mathrm{gz}_2 x = -k'^2 \left\{x + 2\frac{x^3}{3\,!} + 2^3(1+k'^2)\frac{x^5}{5\,!} + 2^4(2+13k'^2+2k'^4)\frac{x^7}{7\,!} + \&c.\right\},$$

$$\mathrm{gz}_3 x = k^2k'^2 \left\{2\frac{x^3}{3\,!} + 2^3(k^2-k'^2)\frac{x^5}{5\,!}\right.$$
$$\left. + 2^4(2k^4 - 13k^2k'^2 + 2k'^4)\frac{x^7}{7\,!} + \&c.\right\}.$$

Since $\qquad \mathrm{iz}_s x = \mathrm{gz}_s x - k^2 x,$

and $\qquad \mathrm{ez}_s x = \mathrm{gz}_s x + k'^2 x,$

the corresponding series for $\mathrm{iz}_s x$ and $\mathrm{ez}_s x$ differ from those for $\mathrm{gz}_s x$ only in the term involving x; the value of this term is

in	$\mathrm{iz}\, x$,	0	in	$\mathrm{ez}\, x$,	x
„	$\mathrm{iz}_1 x$,	$-\tfrac{1}{3}(1+k^2)\,x$	„	$\mathrm{ez}_1 x$,	$\tfrac{1}{3}(1+k'^2)$
„	$\mathrm{iz}_2 x$,	$-x$	„	$\mathrm{ez}_2 x$,	0
„	$\mathrm{iz}_3 x$,	$-k^2 x$	„	$\mathrm{ez}_3 x$,	$k'^2 x.$

The q-series for $Z_s(x)$, § 60.

§ 60. In this paper I have refrained from referring to the q-formulæ which are deducible from some of the groups of equations, but in connexion with the expansions given in the preceding section it seems desirable to give also the q-series for $Z_s(u)$, which are as follows:

$$\left[u = \frac{2Kx}{\pi}\right]$$

$$\frac{2K}{\pi} Z(u) = \sum_1^\infty \frac{4q^n}{1-q^{2n}} \sin 2nx,$$

$$\frac{2K}{\pi} Z_1(u) = \frac{\cos x}{\sin x} + \sum_1^\infty \frac{4q^{2n}}{1-q^{2n}} \sin 2nx,$$

$$\frac{2K}{\pi} Z_2(u) = -\frac{\sin x}{\cos x} + \sum_1^\infty (-)^n \frac{4q^{2n}}{1-q^{2n}} \sin 2nx,$$

$$\frac{2K}{\pi} Z_3(u) = \sum_1^\infty (-)^n \frac{4q^n}{1-q^{2n}} \sin 2nx.$$

The corresponding q-series for

$$\frac{2K}{\pi}\,\mathrm{iz}_s u, \quad \frac{2K}{\pi}\,\mathrm{gz}_s u, \quad \frac{2K}{\pi}\,\mathrm{ez}_s u, \quad \frac{2K}{\pi}\,\zeta_s(u),$$

are deducible from the above series for $\frac{2K}{\pi} Z_s(u)$ by the addition of the terms

$$\frac{4KI}{\pi^2}x, \quad \frac{4KG}{\pi^2}x, \quad \frac{4KE}{\pi^2}x, \quad \frac{2Kx}{\pi K'},$$

respectively.

List of indefinite integrals, § 61.

§ 61. The following is a list of the values of 32 indefinite integrals, expressed by means of the functions $\mathrm{iz}_s x$, $\mathrm{gz}_s x$, $\mathrm{ez}_s x$. This list is extracted from vol. XIX, pp. 146–148 of the

Quarterly Journal. In the list as it appeared in the *Quarterly Journal* the values were all expressed in terms of $Z(x)$.

$$(1)\qquad k'^2 \int \frac{1}{1-\operatorname{sn} x}\, dx = -\operatorname{gz}_2 x + \frac{\operatorname{dn} x}{\operatorname{cn} x},$$

$$(2)\qquad k'^2 \int \frac{1}{1+\operatorname{sn} x}\, dx = -\operatorname{gz}_2 x - \frac{\operatorname{dn} x}{\operatorname{cn} x},$$

$$(3)\qquad k'^2 \int \frac{1}{1-k\operatorname{sn} x}\, dx = \operatorname{ez}_3 x - k\frac{\operatorname{cn} x}{\operatorname{dn} x},$$

$$(4)\qquad k'^2 \int \frac{1}{1+k\operatorname{sn} x}\, dx = \operatorname{ez}_3 x + k\frac{\operatorname{cn} x}{\operatorname{dn} x},$$

$$(5)\qquad \int \frac{1}{1-\operatorname{cn} x}\, dx = -\operatorname{iz}_1 x - \frac{\operatorname{dn} x}{\operatorname{sn} x},$$

$$(6)\qquad \int \frac{1}{1+\operatorname{cn} x}\, dx = -\operatorname{iz}_1 x + \frac{\operatorname{dn} x}{\operatorname{sn} x},$$

$$(7)\qquad k^2 \int \frac{1}{1-\operatorname{dn} x}\, dx = -\operatorname{iz}_1 x - \frac{\operatorname{cn} x}{\operatorname{sn} x},$$

$$(8)\qquad k^2 \int \frac{1}{1+\operatorname{dn} x}\, dx = -\operatorname{iz}_1 x + \frac{\operatorname{cn} x}{\operatorname{sn} x},$$

$$(9)\qquad k^2k' \int \frac{1}{\operatorname{dn} x-k'}\, dx = -\operatorname{gz}_2 x + k'\frac{\operatorname{sn} x}{\operatorname{cn} x},$$

$$(10)\qquad k^2k' \int \frac{1}{\operatorname{dn} x+k'}\, dx = \operatorname{gz}_2 x + k'\frac{\operatorname{sn} x}{\operatorname{cn} x},$$

$$(11)\qquad k'^2 \int \frac{\operatorname{sn} x}{1-\operatorname{sn} x}\, dx = -\operatorname{ez}_2 x + \frac{\operatorname{dn} x}{\operatorname{cn} x},$$

$$(12)\qquad k'^2 \int \frac{\operatorname{sn} x}{1+\operatorname{sn} x}\, dx = \operatorname{ez}_2 x + \frac{\operatorname{dn} x}{\operatorname{cn} x},$$

$$(13)\quad kk'^2 \int \frac{\operatorname{sn} x}{1 - k \operatorname{sn} x}\, dx = \operatorname{gz}_3 x - k\,\frac{\operatorname{cn} x}{\operatorname{dn} x},$$

$$(14)\quad kk'^2 \int \frac{\operatorname{sn} x}{1 + k \operatorname{sn} x}\, dx = -\operatorname{gz}_3 x - k\,\frac{\operatorname{cn} x}{\operatorname{dn} x},$$

$$(15)\quad \int \frac{\operatorname{cn} x}{1 - \operatorname{cn} x}\, dx = -\operatorname{ez}_1 x - \frac{\operatorname{dn} x}{\operatorname{sn} x},$$

$$(16)\quad \int \frac{\operatorname{cn} x}{1 + \operatorname{cn} x}\, dx = \operatorname{ez}_1 x - \frac{\operatorname{dn} x}{\operatorname{sn} x},$$

$$(17)\quad k^2 \int \frac{\operatorname{dn} x}{1 - \operatorname{dn} x}\, dx = -\operatorname{gz}_1 x - \frac{\operatorname{cn} x}{\operatorname{sn} x},$$

$$(18)\quad k^2 \int \frac{\operatorname{dn} x}{1 + \operatorname{dn} x}\, dx = \operatorname{gz}_1 x - \frac{\operatorname{cn} x}{\operatorname{sn} x},$$

$$(19)\quad k^2 \int \frac{\operatorname{dn} x}{\operatorname{dn} x - k'}\, dx = -\operatorname{iz}_2 x + k'\,\frac{\operatorname{sn} x}{\operatorname{cn} x},$$

$$(20)\quad k^2 \int \frac{\operatorname{dn} x}{\operatorname{dn} x + k'}\, dx = -\operatorname{iz}_2 x - k'\,\frac{\operatorname{sn} x}{\operatorname{cn} x},$$

$$(21)\quad kk'^2 \int \frac{\operatorname{cn} x}{\operatorname{dn} x - k \operatorname{cn} x}\, dx = \operatorname{gz} x + k \operatorname{sn} x,$$

$$(22)\quad kk'^2 \int \frac{\operatorname{cn} x}{\operatorname{dn} x + k \operatorname{cn} x}\, dx = -\operatorname{gz} x + k \operatorname{sn} x,$$

$$(23)\quad k'^2 \int \frac{\operatorname{dn} x}{\operatorname{dn} x - k \operatorname{cn} x}\, dx = \operatorname{ez} x + k \operatorname{sn} x,$$

$$(24)\quad k'^2 \int \frac{\operatorname{dn} x}{\operatorname{dn} x + k \operatorname{cn} x}\, dx = \operatorname{ez} x - k \operatorname{sn} x,$$

$$(25)\qquad k'^2 \int \frac{\operatorname{cn} x}{\operatorname{dn} x - \operatorname{cn} x}\, dx = -\operatorname{ez}_1 x - \frac{1}{\operatorname{sn} x},$$

$$(26)\qquad k'^2 \int \frac{\operatorname{dn} x}{\operatorname{dn} x - \operatorname{cn} x}\, dx = -\operatorname{gz}_1 x - \frac{1}{\operatorname{sn} x},$$

$$(27)\qquad k'^2 \int \frac{\operatorname{cn} x}{\operatorname{dn} x + \operatorname{cn} x}\, dx = \operatorname{ez}_1 x - \frac{1}{\operatorname{sn} x},$$

$$(28)\qquad k'^2 \int \frac{\operatorname{dn} x}{\operatorname{dn} x + \operatorname{cn} x}\, dx = -\operatorname{gz}_1 x + \frac{1}{\operatorname{sn} x},$$

$$(29)\qquad k' \int \frac{\operatorname{sn} x}{\operatorname{dn} x - k' \operatorname{sn} x}\, dx = -\operatorname{ez}_2 x + \frac{k'}{\operatorname{cn} x},$$

$$(30)\qquad \int \frac{\operatorname{dn} x}{\operatorname{dn} x - k' \operatorname{sn} x}\, dx = -\operatorname{iz}_2 x + \frac{k'}{\operatorname{cn} x},$$

$$(31)\qquad k' \int \frac{\operatorname{sn} x}{\operatorname{dn} x + k' \operatorname{sn} x}\, dx = \operatorname{ez}_2 x + \frac{k'}{\operatorname{cn} x},$$

$$(32)\qquad \int \frac{\operatorname{dn} x}{\operatorname{dn} x + k' \operatorname{sn} x}\, dx = -\operatorname{iz}_2 x - \frac{k'}{\operatorname{cn} x}.$$

ON A THEOREM IN THE CALCULUS OF VARIATIONS.

By *R. A. Roberts, M.A.*

It is a known theorem in the Calculus of Variations that, if the integral

$$\int \sqrt{\{(X_1 + X_2)(dx_1^2 + dx_2^2)\}} \;\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be made a maximum or a minimum, then the relation between x_1 and x_2 is given by the equation

$$\frac{dx_1}{\sqrt{(X_1 + c)}} \pm \frac{dx_2}{\sqrt{(X_2 - c)}} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where X_1 and X_2 are functions of x_1 and x_2 respectively, and c is a constant. Now an obvious extension of this result to the case of n variables is that, if we substitute $X_1+X_2+\ldots+X_n$ for X_1+X_2, and $dx_1^2+dx_2^2+\ldots+dx_n^2$ for $dx_1^2+dx_2^2$ in (1), the $n-1$ relations connecting x_1, x_2, &c. are given by

$$\frac{dx_1}{\sqrt{(X_1+c_1)}}=\frac{dx_2}{\sqrt{(X_2+c_2)}}=\ldots=\frac{dx_n}{\sqrt{(X_n+c_n)}}\ldots\ldots(3),$$

where c_1, c_2, &c are n constants connected by the equation $\Sigma^1{}_n c=0$. This result, it may be observed, readily follows from the application of the principle of least action to the motion of a particle in space of n dimensions.

I propose, however, to investigate here an extension of (2) of a different kind from that just mentioned. Writing (1) in the form

$$\delta\int\sqrt{\{(X_1-X_2)(dx_1^2-dx_2^2)\}}=0,$$

let us consider the following generalization: To find the relations consequent upon making the $\int du$ a maximum or a minimum, where

$$du^2=f'(X_1)\,dx_1^2+f'(X_2)\,dx_2^2+\ldots+f'(X_n)\,dx_n^2\ldots(4),$$

$f(X)$ being equal to $(X-X_1)(X-X_2)\ldots(X-X_n)$.

Now I intend to shew that these relations are

$$\left.\begin{array}{ll}\Sigma^1{}_n\dfrac{dx_i}{\sqrt{\{\phi(x_i)\}}}=0, & \Sigma^1{}_n\dfrac{X_i dx_i}{\sqrt{\{\phi(x_i)\}}}=0,\\[2ex] \Sigma^1{}_n\dfrac{X_i^2 dx_i}{\sqrt{\{\phi(x_i)\}}}=0, & \Sigma^1{}_n X_i^{n-1}\dfrac{dx_i}{\sqrt{\{\phi(x_i)\}}}=0,\end{array}\right\}\ldots\ldots(5)$$

where $\qquad \phi(x_i)=(X_i+c_1)(X_i+c_2)\ldots(X_i+c_{n-1}),$

c_1, c_2, $\ldots c_{n-1}$ being $n-1$ arbitrary constants introduced by integration. In order to prove this I consider the case of three variables x_1, x_2, x_3, the procedure being exactly the same as in the general case. Performing then the operation δ upon the integral $\int du$, where

$$du^2=(X_1-X_2)(X_1-X_3)\,dx_1^2+(X_2-X_1)(X_2-X_3)\,dx_2^2$$
$$+(X_3-X_1)(X_3-X_2)\,dx_3^2\ldots(6);$$

and equating to zero the coefficients of δx_1, δx_2, δx_3 under

the sign of integration, we obtain

$$\left.\begin{aligned}
&X_1'(2X_1 - X_1 - X_3)\frac{dx_1^2}{du^2} - X_1'(X_2 - X_3)\left(\frac{dx_2^2}{du^2} - \frac{dx_3^2}{du^2}\right) \\
&\qquad\qquad - \frac{2d}{du}\left\{(X_1 - X_2)(X_2 - X_3)\frac{dx_1}{du}\right\} = 0 \\
&X_2'(2X_2 - X_3 - X_1)\frac{dx_2^2}{du^2} - X_2'(X_3 - X_1)\left(\frac{dx_3^2}{du^2} - \frac{dx_1^2}{du^2}\right) \\
&\qquad\qquad - \frac{2d}{du}\left\{(X_2 - X_3)(X_2 - X_1)\frac{dx_2}{du}\right\} = 0 \\
&X_3'(2X_3 - X_1 - X_2)\frac{dx_3^2}{du^2} - X_3'(X_1 - X_2)\left(\frac{dx_1^2}{du^2} - \frac{dx_2^2}{du^2}\right) \\
&\qquad\qquad - \frac{2d}{du}\left\{(X_3 - X_1)(X_3 - X_2)\frac{dx_3}{du}\right\} = 0
\end{aligned}\right\}\ldots(7).$$

From (5) then the integrals of these equations (7) ought to be comprised in

$$\Sigma^1{}_3 \frac{dx_i}{\sqrt{\{(X_i+c_1)(X_i+c_2)\}}} = 0, \quad \Sigma^1{}_3 \frac{X_i dx_i}{\sqrt{\{(X_i+c_1)(X_i+c_2)\}}} = 0\ldots(8),$$

and I shall shew that this is the case. Solving for $\frac{dx_1}{du}, \frac{dx_2}{du}, \frac{dx_3}{du}$ from (6) and (8), we obtain

$$\frac{dx_1}{du} = \frac{\sqrt{\{(X_1+c_1)(X_1+c_2)\}}}{f'(X_1)}, \quad \frac{dx_2}{du} = \frac{\sqrt{\{(X_2+c_1)(X_2+c_2)\}}}{f'(X_2)},$$

$$\frac{dx_3}{du} = \frac{\sqrt{\{(X_3+c_1)(X_3+c_2)\}}}{f'(X_3)},$$

where $\quad f(X) = (X - X_1)(X - X_2)(X - X_3),$

substituting which values in the equations (7) they are found to be identically satisfied. In the same way we arrive at the equations (5) in the general case of n variables.

As an application of the preceding results we may notice that the expression for the element of the arc of a curve in elliptic coordinates comes under the form (6). In fact, using Cayley's notation, if p, q, r are the roots of the equation

$$\frac{x^2}{a+p} + \frac{y^2}{b+p} + \frac{z^2}{c+p} - 1 = 0\ldots\ldots\ldots\ldots.(9),$$

we have

$$4ds^2 = (p-q)(p-r)\,d\alpha^2 + (q-p)(q-r)\,d\beta^2 + (r-p)(r-q)\,d\gamma^2,$$

where

$$d\alpha^2 = \frac{dp^2}{(a+p)(b+p)(c+p)}, \qquad d\beta^2 = \frac{dq^2}{(a+q)(b+q)(c+q)}$$

$$d\gamma^2 = \frac{dr^2}{(a+r)(b+r)(c+r)}.$$

Hence for a line, expressing that $\int ds$ is a minimum, we obtain from (5)

$$\Sigma \frac{d\alpha}{\sqrt{\{(p+c_1)(p+c_2)\}}} = 0, \quad \Sigma \frac{p\,d\alpha}{\sqrt{\{(p+c_1)(p+c_2)\}}} = 0 \ldots (10),$$

which are evidently equivalent to Liouville's differential equations of the system of lines touching two fixed confocal quadrics (see *Liouville's Journal de Mathematiques*, t. XII., p. 418). The integrals involved in (10) are evidently the first kind of hyper-elliptic integrals; and in the same way by making use of the extension of elliptic coordinates to the case of n dimensions we obtain the general system of hyper-elliptic integrals with the corresponding algebraic relations.

It may be observed that the two equations (8) are unaltered if we substitute for X_1, X_2, X_3 a homographic transformation of these quantities, excepting, of course, that the constants are transformed in the same way. It thus appears that we obtain the same integrals of the equations resulting from making $\delta \int du = 0$ as from the more general condition $\delta \int dr = 0$, where

$$dr^2 = (X_1+\alpha)^{-1}(X_2+\alpha)^{-1}(X_3+\alpha)^{-1}\left\{\frac{(X_1-X_2)(X_1-X_3)\,dx_1^2}{(X_1+\alpha)}\right.$$

$$\left. + \frac{(X_2-X_3)(X_2-X_1)\,dx_2^2}{(X_2+\alpha)} + \frac{(X_3-X_1)(X_3-X_2)\,dx_3^2}{(X_3+\alpha)}\right\} \ldots (11),$$

α being a constant. This result will evidently also hold good in the general case, when the number of the variables is odd.

If we apply (11) to the case we have already considered in elliptic coordinates, we find that a line is the only curve which satisfies the condition

$$\delta \int \sqrt{\left\{\frac{\cos^2 i}{p+\alpha} + \frac{\cos^2 i'}{q+\alpha} + \frac{\cos^2 i''}{r+\alpha}\right\}} \frac{ds}{\sqrt{\{(p+\alpha)(q+\alpha)(r+\alpha)\}}} = 0 \ldots (12),$$

where i, i', i'' are the angles which the tangent to the curve at a point P makes with the normals to the three confocals which pass through P. This result (12) may be stated in a more geometrical form, as follows: Let r, r' be the portions of the tangent to a curve measured from the point of contact to the points of intersection with a given quadric, then if

$$\delta \int \left(\frac{1}{r} - \frac{1}{r'}\right) ds = 0,$$

the curve must be a right line.

ON PLANE CUBICS SATISFYING CERTAIN CONDITIONS.

By *R. A. Roberts, M.A.*

THE problem considered in this note is to describe a plane cubic passing through three collinear points and touching pairs of lines intersecting in these points. The interest of this problem lies in the fact that the equations of the cubics which satisfy the conditions can be easily obtained. Let u be the line on which the three points lie and let x, y, z be the fourth harmonics to u and the pairs of tangents, then these tangents may be written

$$u^2 - x^2 = 0, \quad u^2 - y^2 = 0, \quad u^2 - z^2 = 0 \text{.........} (1),$$

where we suppose $u = ax + by + cz$. Now, if we suppose the first two pairs corresponding tangents of the same system, it is easy to see that the points of contact will lie on a conic passing through ux, uy. In this case the equation

$$(u^2 - x^2)(u^2 - y^2) - V^2 = 0 \text{..............} (2)$$

being divided by u will represent the curve, the conic V being supposed to be of the form $uL + xy$; but since the curve passes through uz, we have

$$V = u(lu + mz) + xy \text{.................} (3).$$

Expressing then that the curve

$$(u^2 - x^2)(u^2 - y^2) - \{u(lu + mz) + xy\}^2 = 0 \text{......} (4)$$

touches the lines $u \pm z = 0$, we get

$(l+m)^2 - 1 = 0 \ldots (5)$, or $a^2 + b^2 + 2(l+m)ab = (c-1)^2 \ldots (6)$,

and

$(l-m)^2 - 1 = 0 \ldots (7)$, or $a^2 + b^2 + 2(l+m)ab = (c+1)^2 \ldots (8)$.

The conditions (5) and (7) taken together give the cubics

$$u^3 \pm 2xyz - u(x^2 + y^2 + z^2) = 0 \ldots\ldots\ldots\ldots\ldots (9),$$

and (6) and (8) give a cubic which, multiplied by u, can be written in the form

$$a\sqrt{(u^2 - x^2)} + b\sqrt{(u^2 - y^2)} + c\sqrt{(u^2 - z^2)} = 0 \ldots\ldots (10);$$

also (5), (8) and (6), (7) give four cubics.

These latter cubics are not symmetrical with regard to x, y, z, and therefore, by interchanging the variables, we obtain eight other similar curves; but the three cubics (9) and (10) are symmetrical. We thus have fifteen cubics altogether satisfying the given conditions.

In the case of the two cubics (9) the six points of contact with the lines (1) lie by threes on four lines; for, writing

$$u \pm (y + z - x) = \alpha, \qquad u \pm (z + x - y) = \beta,$$
$$u \pm (x + y - z) = \gamma, \qquad u \mp (x + y - z) = \delta,$$

the equation (9) becomes

$$\alpha\beta\gamma + \delta\beta\gamma + \delta\gamma\alpha + \delta\alpha\beta = 0 \ldots\ldots\ldots\ldots (11);$$

and the tangents at $\alpha\beta$, $\alpha\gamma$, &c. are then $\alpha + \beta = 0$, $\alpha + \gamma = 0$, &c. or $u \pm z = 0$, $u \pm y = 0$, namely, the lines (1), which proves what we have stated above.

The cubic (10) is evidently the only absolutely unique cubic of the system. For this cubic and the two cubics (9) the three pairs of lines (1) are corresponding tangents of the same system, and for the twelve other cubics two pairs of the lines are corresponding tangents of the same system, the other pair belonging to a different system.

It might appear that there ought to be cubics corresponding to the case of the three pairs of lines being tangents of the three different systems; but there cannot be such curves unless the lines (1) are connected by a certain relation, in which case there are an infinite number of cubics satisfying the given conditions. From Clebsch's representation of points on the curve by means of elliptic integrals we can easily shew

that in the case we are considering the points of contact of the tangents will lie on two right lines. If S is the equation of the curve we may write then

$$\left.\begin{array}{l}(u-x)(u-y)(\bar{u}-z)-u\delta^2 = \quad S=0 \\ (u+x)(u+y)(u+z)-u\delta'^2 = -S=0\end{array}\right\}\text{........}(12).$$

Hence we have identically

$$2(u^2+xy+yz+zx) \equiv \delta^2+\delta'^2,$$

from which we see that the discriminant of the left-hand side vanishes, or

$$a^2+b^2+c^2-2ab-2bc-2ca=1\text{.........}(13),$$

and the corresponding system of cubics may be written

$$u(\lambda^2P^2+Q^2)+2\lambda\{u^2(x+y+z)+xyz\}=0\text{...}(14),$$

where $$u^2+xy+yz+zx=PQ,$$

and λ is a variable parameter.

If the six lines (1) are all touched by a conic the three lines x, y, z will evidently pass through a point, namely, the pole of u with regard to the conic. In this case the cubic (10) becomes irrelevant, and for the cubics (9) it is easy to see that the pole of u with regard to the conic is one of its poles with regard to the curve; also, if we substitute

$$ax+\frac{y}{a}, \quad bx+\frac{y}{b}, \quad cx+\frac{y}{c},$$

for x, y, z, respectively, the system (14) may be written

$$rz\cos\vartheta\{z^2-(mx+ny)^2\}+2rz(mx+ny)\sin\vartheta$$
$$+r^2x\{(x-n^2y)^2+n^2z^2\}+y\{(y-m^2x)^2+m^2z^2\}=0\text{...}(15),$$

where $a+b+c=n^2abc, \quad ab+bc+ca=m^2, \quad abc=r,$

the quantities a, b, c being connected by the relation

$$(a+b+c)(ab+bc+ca)=9abc, \text{ or } mn=3,$$

and ϑ is a variable parameter.

Let us suppose two of the points on the line u to be the circular points at infinity, then the problem which we have been considering evidently becomes: to describe a circular cubic, being given two foci and two parallel lines as tangents at centres of inversion; for the tangents at these latter points are, as is known, parallel to the real asymptote of the curve.

The fifteen cubics which we have found above constitute the solution in the case of the foci lying on the same J circle (see Dr. Casey's Memoir on Bicircular Quartics). When the foci lie on different J circles, these points and the two parallel lines are connected by a certain relation, namely, $p\varpi + c^2 = 0$, where p is the perpendicular from one focus on one of the lines, ϖ is the perpendicular from the other focus on the other line, and c is half the distance between the foci. In the case considered there are, of course, an infinite number of circular cubics satisfying the given conditions.

A GEOMETRICAL THEOREM.

By *R. Lachlan, B.A.*, Fellow of Trinity College, Cambridge.

THE theorem, which is the subject of this paper, was obtained by generalising the method employed in Salmon's Conics to prove Pascal's theorem, (see Salmon, § 267), and so might have been described as a generalisation of Pascal's theorem; an analogous theorem to Steiner's is also deduced, and a few special cases which seemed worthy of notice are added.

1. Given any conic S, let six other conics u, v, w, u', v', w' be drawn so that the common points of each of the pairs (u, u'), (v, v'), (w, w') lie on the given conic S, so that we have

$$\left.\begin{aligned} S &\equiv \lambda u - \lambda' u' \\ S &\equiv \mu v - \mu' v' \\ S &\equiv \nu w - \nu' w' \end{aligned}\right\},$$

whence we have at once

$$\mu v - \nu w \equiv \mu' v' - \nu' w'.$$

Hence the four common points of (v, w), and the four common points of (v', w') lie on another conic, S_1 say,

where $$S_1 \equiv \mu v - \nu w.$$

Similarly the common points of the pairs (w, u) (w', u') lie on another conic

$$S_2 \equiv \nu w - \lambda u$$

and the common points of the pairs (u, v) (u', v') on a conic

$$S_3 \equiv \lambda u - \mu v.$$

Moreover we see that

$$S_1 + S_2 + S_3 \equiv 0,$$

or the conics S_1 S_2 S_3 have four common points.

Thus the twelve points of intersection of the conics (u, v, w) and the twelve points of intersection of the conics (u', v', w') lie on three conics S_1 S_2 S_3 which have four common points.

2. We see at once that if any pair, say (v, w), represent two pairs of straight lines, one being common to each, then this straight line is part of the locus given by S_1.

Also, if (v, w) be each circles, S_1 is a circle.

If (u, v, w) be each circles, S_1, S_2, S_3 are a system of coaxial circles; if the former are coaxial then S_1, S_2, S_3 belong to the same system of coaxial circles.

3. Reciprocating, we have a theorem analogous to Brianchon's. If the four common tangents of each of the pairs of conics (u, u'), (v, v'), (w, w') are also tangents of a conic S, then the twelve common tangents of the group of conics (u, v, w) and those of the group (u', v', w') touch, in sets of eight, three conics S_1 S_2 S_3, which are all inscribed in the same quadrilateral.

4. Suppose now we have five given points $(1, 2, 3, 4, 5)$; since a conic can always be drawn through them then by taking

$$u \equiv (12)(45), \quad v \equiv (12)(53), \quad w \equiv (12)(34),$$

we see that if conics be drawn through the points $(1, 2, 4, 5)$, $(1, 2, 5, 3)$, $(1, 2, 3, 4)$ respectively, they will intersect in three other points P, Q, R say; then the straight lines connecting these to the points $(3, 4, 5)$ respectively meet in a point.

Or, again, as a particular case, we see that the lines connecting the points $(3, 4, 5)$ respectively to in the intersections of the diagonals of the quadrilaterals $(1, 2, 4, 5)$, $(1, 2, 5, 3)$ and $(1, 2, 3, 4)$ meet in a point. Hence we can deduce at once that the fifteen lines connecting each of five points to the points of intersection of the diagonals of the quadrilaterals formed by joining the remaining four points, intersect one another in twelve points, *i.e.* through each point pass three of the lines.

5. Again, if the points (1, 2) be the circular points at infinity, we have the theorem that if two segments of circles be drawn on each side of a triangle ABC, and intersecting one another in the points P, Q, R, P', Q', R', then the circles circumscribing the triangles (A, P, P'), (B, Q, Q') (C, R, R') have a common radical axis.

6. Suppose that we have six points (1, 2, 3, 4, 5, 6) on a conic, then taking

$$u \equiv (34)(56), \quad v \equiv (56)(12), \quad w \equiv (12)(34),$$

and u', v', w' any conics through the points (3, 4, 5, 6), (5, 6, 1, 2), (1, 2, 3, 4) respectively; if the lines (12, (34), (56) meet in the points A, B, C,

v', w' have a common chord which passes through A,

w', u' „ „ „ „ B,

u', v' „ „ „ „ C,

and these three straight lines meet in a point.

Pascal's and Steiner's theorem are deduced at once from this by taking u', v', w' as pairs of straight lines.

7. The reciprocal of this follows at once from §3. If ABC be any triangle, and tangents (1, 2, 3, 4, 5, 6) be drawn from the angular points to any given conic; and if conics (u, v, w) be drawn to touch the lines (3, 4, 5, 6), (5, 6, 1, 2) (1, 2, 3, 4) respectively; then the other two common tangents to each pair of conics (v, w), (w, u), (u, v) meet in three points lying on the sides BC, CA, AB of the triangle; and these three points lie on a straight line.

For instance, suppose two tangents TP, TQ to be drawn to any conic whose foci are S, S', then if a conic v be drawn to touch TP, TQ, and having S for a focus, and a conic w be drawn with S' for a focus and touching TP, TQ, the other common tangents to v, w meet in a point, L say, lying on SS'. (If two parabolas, foci S, S', be drawn to touch TP, TQ then the other common tangent is parallel to SS'.) Again, if u be any other conic confocal with the given conic, then the common tangents of u, v intersect in a point (M say) on $S'T$; and the common tangents of u, w intersect in a point (N say) on ST, then these three points L, M, N are collinear.

8. From § 2, we see that if v, w be any two circles, and conics v', w' be drawn respectively through the intersections of v, w with any conic S, then the four points of intersection of v', w' lie on a circle which is coaxial with v, w.

If then the circles of curvature at two points Q, R of a conic cut the conic again in q, r, any conics osculating the given conic at Q, R and passing through q, r intersect in four points which lie on a circle coaxial with the circles of curvature at Q, R.

9. Again, if the circles of curvature at three points P, Q, R cut the conic again in p, q, r, then the common points of three conics osculating the given conic at P, Q, R and passing respectively through p, q, r lie on three coaxial circles each of which is coaxial with two of the three circles of curvature.

Thus if the tangents at P, Q, R meet in A, B, C, and if Pp, Qq, Rr meet in A', B', C'; also if Pp, meet AB, AC in D, D'; if Qq meet BC, BA in E, E'; and Rr meet CA, CB in F, F', then the points (A, A', E', F), (B, B', F', D) and (C', C', D', E) lie respectively on three circles which are coaxial; and each of these circles is coaxial with a pair of the three circles of curvature at P, Q, R.

A SYMBOLICAL PROOF OF FOURIER'S DOUBLE-INTEGRAL THEOREM.

By *P. Alexander, M.A.*

In vol. III. pp. 288–290 of the *Cambridge Mathematical Journal* there is a novel and interesting attempt at a proof of Fourier's theorem by a contributor who signs himself G.

He does not seem to be aware that the result he has obtained is only half of what it should be. He shews that

$$\int_0^\infty db \int_0^\infty da f(a) \cos b(a-x) = \tfrac{1}{2}\pi f(x) \text{ instead of } \pi f(x).$$

I attribute G's failure to the use he makes of vague considerations of grades of infinity.

Taking a hint from G's attempt and making the same departure, I have, by a different course from his, obtained the correct result.

My proof is as follows:—

By integration by parts,

$$\int da f(a) \cos b(a-x) = f(a) \frac{\sin b(a-x)}{b} + f'(a) \frac{\cos b(a-x)}{b^2}$$

$$- f''(a) \frac{\sin b(a-x)}{b^3} - f'''(a) \frac{\cos b(a-x)}{b^4}$$

$$+ f^4(a) \frac{\sin b(a-x)}{b^5} + f^5(a) \frac{\cos b(a-x)}{b^6}$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{\sin b(a-x)}{b}\left[f(a) - \frac{f''(a)}{b^2} + \frac{f^4(a)}{b^4} - \ldots\right]$$

$$+ \frac{\cos b(a-x)}{b^2}\left[f'(a) - \frac{f'''(a)}{b^2} + \frac{f^5(a)}{b^4} - \ldots\right]$$

$$= \frac{\sin b(a-x)}{b}\left[1 - \left(\frac{1}{b}\frac{d}{da}\right)^2 + \left(\frac{1}{b}\frac{d}{da}\right)^4 - \ldots\right] f(a)$$

$$+ \frac{\cos b(a-x)}{b^2}\left(\frac{d}{da}\right)\left[1 - \left(\frac{1}{b}\frac{d}{da}\right)^2 + \left(\frac{1}{b}\frac{d}{da}\right)^4 - \ldots\right] f(a)$$

$$= \left[\frac{\sin b(a-x)}{b} + \frac{\cos b(a-x)}{b^2}\left(\frac{d}{da}\right)\right] \frac{f(a)}{1 + \left(\frac{1}{b}\frac{d}{da}\right)^2}$$

$$= \left[b \sin b(a-x) + \cos b(a-x)\left(\frac{d}{da}\right)\right] \frac{f(a)}{b^2 + \left(\frac{d}{da}\right)^2}$$

$$= \left[b \sin bz + \cos bz \left(\frac{d}{da}\right)\right] \frac{f(a)}{b^2 + \left(\frac{d}{da}\right)^2} \ldots\ldots\ldots\ldots (\mathrm{I}),$$

where $$z = a - x.$$

But

$$\int_0^\infty \frac{\cos rx}{1+x^2}\,dx = \tfrac{1}{2}\pi e^{\mp r},$$

$$\int_0^\infty \frac{x \sin rx}{1+x^2} \quad = \pm\,\tfrac{1}{2}\pi e^{\mp r},$$

according as r is positive or negative.

Putting $\dfrac{b}{D}$ for x and zD for r, these become

$$\int_0^\infty \frac{D\cos bz}{b^2+D^2}\,db = \tfrac{1}{2}\pi e^{\mp zD},$$

$$\int_0^\infty \frac{b\sin bz}{b^2+D^2}\,db = \pm\,\tfrac{1}{2}\pi e^{\mp zD},$$

according as z is positive or negative.

From these we may infer the symbolical forms

$$\left.\begin{aligned}\int_0^\infty db\,\frac{\left(\dfrac{d}{da}\right)\cos bz}{b^2+\left(\dfrac{d}{da}\right)^2}f(a) &= \tfrac{1}{2}\pi e^{\mp z\frac{d}{da}}f(a)\\ &= \tfrac{1}{2}\pi f(a\mp z)\\ \int_0^\infty db\,\frac{b\sin bz}{b^2+\left(\dfrac{d}{da}\right)^2}f(a) &= \pm\,\tfrac{1}{2}\pi e^{\mp z\frac{d}{da}}f(a)\\ &= \pm\,\tfrac{1}{2}\pi f(a\mp z)\end{aligned}\right\}\text{......(II)},$$

according as z is positive or negative.

Hence

$$\left.\begin{aligned}\int_0^\infty db\left[\frac{b\sin bz+\left(\dfrac{d}{da}\right)\cos bz}{b^2+\left(\dfrac{d}{da}\right)^2}f(a)\right] &= (\tfrac{1}{2}\pi\pm\tfrac{1}{2}\pi)f(a\mp z)\\ &= \pi f(a-z)\text{ or }0\end{aligned}\right\}\text{...(III)},$$

according as z is positive or negative.

Wherefore, from (I) and (III),

$$\int_0^\infty db \int_\lambda^l da f(a) \cos b(a-x)$$

$$= \int_0^\infty db \left[\frac{b \sin bz + \left(\frac{d}{da}\right) \cos bz}{b^2 + \left(\frac{d}{da}\right)^2} f(a) \right]_{a=\lambda}^{a=l}$$

$$= \int_0^\infty db \left[\frac{b \sin bz + \left(\frac{d}{da}\right) \cos bz}{b^2 + \left(\frac{d}{da}\right)^2} f(a) \right]_{a=l}$$

$$- \int_0^\infty db \left[\frac{b \sin bz + \left(\frac{d}{da}\right) \cos bz}{b^2 + \left(\frac{d}{da}\right)^2} f(a) \right]_{a=\lambda} \quad \ldots (\text{IV}),$$

where $z = a - x$, and $\left(\frac{d}{da}\right)$ operates only on $f(a)$.

Now, if $a = l > x$, z will be positive, therefore

$$\left\{\int_0^\infty db \left[\frac{b \sin bz + \left(\frac{d}{da}\right) \cos bz}{b^2 + \left(\frac{d}{da}\right)^2} f(a) \right]\right\}_{a=l>x} = \pi f(a-z) = \pi f(x).$$

If $a = \lambda < x$, z will be negative,

$$\left\{\int_0^\infty db \left[\frac{b \sin bz + \left(\frac{d}{da}\right) \cos bz}{b^2 + \left(\frac{d}{da}\right)^2} f(a) \right]\right\}_{a=\lambda<x} = 0.$$

from (III).

Therefore (IV) becomes

$$\int_0^\infty db \int_\lambda^l da f(a) \cos b(a-x) = \pi f(x) \quad \ldots\ldots\ldots\ldots (\text{V}),$$

if $l > x > \lambda$.

Similarly, if $x > l > \lambda$, (IV) becomes

$$\int_0^\infty db \int_\lambda^l da f(a) \cos b(a-x) = 0 - 0 = 0 \quad \ldots\ldots (\text{VI});$$

and, if $l > \lambda > x$, (IV) becomes

$$\int_0^\infty db \int_\lambda^l da f(a) \cos b(a-x) = \pi f(x) - \pi f(x) = 0 \;\ldots\ldots\text{(VII)}.$$

Hence, from (V), (VI), and (VII),

$$\int_0^\infty db \int_\lambda^l da f(a) \cos b(a-x) = \pi f(x) \text{ or } 0 \;\ldots\text{(VIII)},$$

according as x lies within or without the limits l and λ.

If $x = a = l$, $z = a - x = 0$, therefore

$$\int^\infty db \left[\frac{b \sin b + \left(\frac{d}{da}\right)\cos bz}{b^2 + \left(\frac{d}{da}\right)^2} f(a)\right]_{a=l=x}$$

$$= \int_0^\infty db \left[\frac{\left(\frac{d}{da}\right)}{b^2 + \left(\frac{d}{da}\right)^2} f(a)\right]_{a=l=x}$$

$$= \left[\tan^{-1}\left(\frac{b}{\frac{d}{da}}\right) f(a)\right]_{b=0}^{b=\infty} = \tfrac{1}{2}\pi f(a);$$

or, if this be looked upon as doubtful, (II) will give

$$\int_0^\infty db \left[\frac{\left(\frac{d}{da}\right)}{b^2 + \left(\frac{d}{da}\right)^2} f(a)\right]_{a=l=x}$$

$$= \int_0^\infty db \left[\frac{\frac{d}{da}\cos bz}{b^2 + \left(\frac{d}{da}\right)^2} f(a)\right]_{a=l=x}$$

$$= \tfrac{1}{2}\pi\, e^{\mp z\left(\frac{d}{da}\right)} f(a) = \tfrac{1}{2}\pi f(a),$$

since $$z = 0;$$

therefore

$$\int_0^\infty db \left[\frac{b\sin bz + \left(\dfrac{d}{da}\right)\cos bz}{b^2 + \left(\dfrac{d}{da}\right)^2} f(a)\right]_{a=l=x} = \tfrac{1}{2}\pi f(a) = \tfrac{1}{2}\pi f(x).$$

Also, by (III),

$$\int_0^\infty db \left[\frac{b\sin bz + \left(\dfrac{d}{da}\right)\cos bz}{b^2 + \left(\dfrac{d}{da}\right)^2} f(a)\right]_{a=\lambda<x} = 0;$$

therefore (IV) becomes

$$\left.\begin{aligned}\int_0^\infty db \int_\lambda^l da f(a)\cos b(a-x) &= \tfrac{1}{2}\pi f(x) - 0\\ &= \tfrac{1}{2}\pi f(x)\end{aligned}\right\}\text{.....(IX)},$$

if $x = l$.

Similarly, it may be shown that

$$\int_0^\infty db \int_\lambda^l da f(a)\cos b(a-x) = \tfrac{1}{2}\pi f(x)\text{.........(X)},$$

if $x = \lambda$.

Therefore, from (VIII), (IX), and (X), it follows that

$$\int_0^\infty db \int_\lambda^l da f(a)\cos b(a-x) = \pi f(x),\ \tfrac{1}{2}\pi f(x),\text{ or } 0\text{...(XI)},$$

according as x lies within, upon, or without the limits l and λ.

From this it follows that

$$\int_0^\infty db \int_0^{l=\infty} da f(a)\cos b(a-x) = \pi f(x),\ \tfrac{1}{2}\pi f(x),\text{ or } 0\text{...(XII)},$$

according as x is positive, zero, or negative.

From (XI) it also follows that

$$\int_0^\infty db \int_{\lambda=-\infty}^{l=\infty} da f(a)\cos b(a-x) = \pi f(x)\text{......(XIII)},$$

for all values of x.

It is worthy of remark that we have arrived at (XII) and (XIII) by putting

$$l = \infty \text{ and } \lambda = 0 \quad \text{in (XII)},$$
$$\text{and } l = \infty \text{ and } \lambda = -\infty \text{ in (XIII)},$$

after both integrations, but there does not appear to me to be anything to hinder these substitutions after the first integration, except when

$$\int_0^\infty da f(a) \cos b(a-x), \text{ or } \int_{-\infty}^\infty da f(a) \cos b(a-x)$$

become *indeterminate or infinite* by this substitution, when the substitution *must* be deferred till after the second integration.

Furthermore, the order of integration cannot be inverted, because in that case the integral $\int_0^\infty db \cos b(a-x)$ would require to be evaluated, which is indeterminate unless evaluated by Poisson's method of introducing a multiplier e^{-Kb} and putting $K=0$ after *both* integrations, which procedure *unnecessarily* complicates investigations where Fourier's Theorem is employed.

A beautifully simple proof of (XI) is given in Todhunter's Integral Calculus, §§ 328–331.

The proof given in § 332 and in many other books, said to be due to Poisson, gives

$$\phi(x) = \frac{1}{\pi} \int_0^\infty du \int_{-\infty}^\infty dv\, \phi(v) \cos u(v-x),$$

where, from the nature of the proof, the limits of v must be put $=\infty$ and $-\infty$, *immediately after the integration with respect to* v, and it therefore applies only to functions which are such that

$$L_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{l} \phi(v)\, dv\right] = 0;$$

and is therefore not perfectly general, leaving out as it does such important functions as

$$\phi(x) = t_1 \text{ for positive values of } x,$$
$$= t_2 \quad ,, \quad \text{negative} \quad ,, \quad ,,$$
$$\phi(x) = x \text{ for all values of } x,$$
$$\phi(x) = t \, . \, e^{mx},$$

where t, t_1, t_2, and m are constants.

NOTE ON WEIERSTRASS'S THEORY OF DOUBLY PERIODIC FUNCTIONS.

By *A. R. Forsyth.*

WITHIN the last few years many investigations, founded on Weierstrass's method in the Theory of Elliptic Functions, have been published; and rather more than three years ago there appeared the first instalment of what promises to be a full abstract of Weierstrass's lectures on this particular subject, under the title *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen.* An additional reason which increases, if possible, the interest of this work is that the doubly periodic functions which enter are such as to have their periods independent of one another, instead of being functions of a single quantity as are the $4K$ and $4iK'$ in the Jacobian theory of elliptic functions. But (as its title implies) the work is not a full development of the theory; it is, in large part, a statement of results. After the earlier sections the results are by no means difficult to obtain; but the same remark does not apply to these earlier sections in which the most important and fundamental formulæ of the theory are given.

It seems desirable that some account in English should exist so as to be within convenient reach of readers, to prove sufficient to lead them to the fundamental formulæ spoken of and so to enable them to follow the theory in its subsequent developments. The only account of this kind with which I am acquainted is that given by Mr. A. L. Daniels in the *American Journal of Mathematics*, vol. VI., and is contained in two notes. The second is the more important; in it he derives the results by taking as his starting-point the general theory of functions, following in this respect Weierstrass himself. In the present paper the results desired are obtained in what seems to me a slightly different way, founded on the theory of periodic functions as given in Liouville's lectures (*Crelle*, t. LXXXVIII., pp. 277–310) and in Briot et Bouquet, *Théorie des fonctions elliptiques.* No claim for originality is here made; it is essentially a collection and reproduction, with occasional applications, of materials which will be found in the two authorities already quoted, in the lectures of Weierstrass and in memoirs by Kiepert (*Crelle*, t. LXXVI) and Simon (*Crelle*, t. LXXXI). And nothing more

than an introduction is attempted, because when the point indicated is once reached the remaining investigations can be read in, or worked out with comparative ease from, the *Formeln und Lehrsätze.*

Introduction; some general theorems relating to functions.

1. It is necessary to premise certain results from the theory of functions which are subsequently applied.

The variable z is of the complex form $x+iy$; the positive value of $(x^2+y^2)^{\frac{1}{2}}$ is called its modulus. Any function of z, as $f(z)$, will also in general be of the complex form as $X+iY$; it can vanish only when X and Y both vanish. It is convenient to use the ordinary geometrical representation of the variable z by a point in a plane, whose coordinates are x and y; this point is spoken of as the point z. As the variable changes, the point z moves in the plane.

I. If $\phi(z)$ be a function of z which for points within any given area in the plane is everywhere one-valued, finite and continuous, and if t be a point within that area, then

$$0=\int\phi(z)\,dz \quad \text{(i)},$$

$$\phi(t)=\frac{1}{2\pi i}\int\frac{\phi(z)}{z-t}\,dz \quad \text{(ii)},$$

$$\frac{d^n\phi(t)}{dt^n}=\frac{n!}{2\pi i}\int\frac{\phi(z)}{(z-t)^{n+1}}\,dz \quad \text{(iii)},$$

the integrals being in each case taken round the linear boundary of the area under consideration.*

II. If $\psi(z)$ be an one-valued function which is infinite for several points within the area but not for any point on the boundary of the area, and if with these points as centres infinitesimal circles be described, then the value of $\int\phi(z)\,dz$ taken round the area is equal to the sum of the values of $\int\phi(z)\,dz$ taken in the same direction round each of the circles.†

III. When an one-valued function is finite for a value a of z and is finite and continuous for values of z such that the modulus of $z-a$ is less than a quantity R, then in the

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* Briot et Bouquet, *Théorie des fonctions elliptiques*, §§ 82, 85, 86. This treatise will in future be referred to as *B.*; Liouville's lectures in *Crelle*, t. LXXXVIII. as *L.*

† B. p. 195.

neighbourhood of the point α, as defined by a circle of radius R and centre α, the function is expansible in a converging series of ascending powers of $z-\alpha$.*

IV. Hence if an one-valued function $\phi(z)$ have, for $z=\alpha$, an infinite value of order n and for no other point in the immediate neighbourhood of α have an infinite value, then it is possible to choose $G_1, G_2, \ldots, G_n$, so that

$$\phi(z)-\frac{G_1}{z-\alpha}-\frac{G_2}{(z-\alpha)^2}-\ldots-\frac{G_n}{(z-\alpha)^n}$$

is finite in the immediate neighbourhood of $z-\alpha$. For, by definition of the infinity of order n, the limiting value of $(z-\alpha)^n\phi(z)$ is finite, say, G_n, and therefore, by II, $(z-\alpha)^n\phi(z)$ is expansible in the form

$$G_n+G_{n-1}(z-\alpha)+\ldots+G_1(z-\alpha)^{n-1}+G(z-\alpha)^n+\ldots$$

from which the result at once follows.†

Hence if such a function $\phi(z)$ have only a single infinity, and that infinity be of order n, it is possible to choose $G_1, G_2, \ldots, G_n$, so that

$$\phi(z)-\frac{G_1}{z-\alpha}-\frac{G_2}{(z-\alpha)^2}-\ldots-\frac{G_n}{(z-\alpha)^n}$$

is everywhere finite.

Now by I we have

$$\int\frac{dz}{z-\alpha}=2\pi i,$$

$$\int\frac{dz}{(z-\alpha)^m}=0,\ (m>1),$$

if taken round any area enclosing the point α; hence, by II, we have for the last function

$$\int\phi(z)\,dz=2\pi G_1 i,$$

the integral being taken round any area which encloses the point α.

V. If an one-valued function $\phi(z)$ have n infinite values for the points $\alpha_1, \alpha_2, \ldots, \alpha_n$, each of the first order, and have

* B. p. 149.

† These and subsequent propositions are derived from B and L.

no other infinite values, it is possible to choose $G_1, G_2, \ldots, G_n$, so that

$$\phi(z) - \frac{G_1}{z-\alpha_1} - \frac{G_2}{z-\alpha_2} - \ldots - \frac{G_n}{z-\alpha_n}$$

is everywhere finite.

In the same way as in IV we find that

$$\int \phi(z)\, dz = 2\pi i\,(G_1 + G_2 + \ldots + G_n)$$

taken round the boundary of an area which encloses the points $\alpha_1, \alpha_2, \ldots, \alpha_n$.

VI. If a function be one-valued finite and continuous, and its modulus be always less than some fixed finite quantity, then the function can only be a constant.

Let M be the modulus of $f(z)$; since M is never greater than some fixed quantity, it must have a maximum value, say M_0, for a variable z_0. Then we must have $M^2 - M_0^2$, that is,

$$f(x+iy)\,f(x-iy) - f(x_0+iy_0)\,f(x_0-iy_0),$$

negative for all values of x any y. But if we take $x = x_0 + \varepsilon h$ and $y = y_0 + \varepsilon k$ it can be proved (as in Todhunter's *Theory of Equations*, § 29) that h and k may be so chosen that the sign of $M^2 - M_0^2$ is not always the same. As these contradictory results follow from the assumption of variability in the function, it follows that the function, under the assumed conditions, can only be a constant.

VII. A function, which is one-valued finite and continuous, and is not a constant, must have infinite values.

For $M^2 - M_0^2$ can be made positive when ε is small, hence the modulus can be made to increase continually and the function must, for some value of z, be infinite.

2. When we deal with doubly periodic functions a convenient geometrical representation of a set of values of z, which give all the values of the function, can be given. Let $\omega = w_1 + iv_1$ and $\omega' = w_2 + iv_2$ be the periods of a function, w_1, w_2, v_1, v_2 being supposed real. Let points $A_1, A_2, A_3, \ldots$ (fig. 4) represent $\omega, 2\omega, 3\omega, \ldots$ respectively, and points $B_1, B_2, B_3, \ldots$ represent $\omega', 2\omega', 3\omega', \ldots$ respectively; these two sets of points lie on two straight lines through the origin O. Through each of the points let a line be drawn parallel to that on which the other set lies; the plane will thus be divided into a series of parallelograms, and any point will lie

in or on the boundary of some parallelogram. Let a point P, representing the value z, lie in the parallelogram $OA_1C_1B_1$; and let $P_1, P_2, \ldots, Q_1, Q_2, \ldots, R_1, \ldots$ be points in the other parallelograms, each situated in its own similarly to P in $OA_1C_1B_1$. These points represent $z + m\omega + n\omega'$ for the combinations of integral values of m and n.

Now, when a function $\phi(z)$ is doubly periodic in ω and ω', we have

$$\phi(z + m\omega + n\omega') = \phi(z),$$

and therefore the value of the function for each of these points is the same as that for P. Hence it follows that, so far as the values of $\phi(z)$ are concerned, it is sufficient to consider points z lying within or on the boundary of one parallelogram. These values of z are called irreducible values.

3. We now have the following propositions relating to doubly periodic functions.

VIII. A function of z which is one-valued, doubly periodic and continuous, must have infinite values for irreducible values of the variable.

This follows at once from VII and the preceding explanation in §2.

Let such a function being denoted by $\phi(z)$.

IX. The value of $\int\phi(z)\,dz$ taken round the boundary of the parallelogram is zero. For, at two points p and q in the sides OA_1 and C_1B_1 such that pq is parallel to OB_1, the value of $\phi(z)$ is the same, while the elements dz at those points are equal in magnitude but opposite in sign. By taking opposite elements in this way from the two pairs of sides we can make the whole sum disappear.

X. When $\phi(z)$ has a single infinity of order n for $z = \alpha$, then

$$\phi(z) - \frac{G_2}{(z-\alpha)^2} - \frac{G_3}{(z-\alpha)^3} - \ldots - \frac{G_n}{(z-\alpha)^n}$$

is finite everywhere within the parallelogram.

This follows from IV and IX.

XI. A function $\phi(z)$ with only a single infinity of the first order cannot exist as a function; it is a constant.

This follows from VI and X.

XII. A function $\phi(z)$ which has n simple irreducible infinities, and is expressed as in V, is such that

$$G_1 + G_2 + \dots + G_n = 0.$$

This follows from IX.

Hence a function which has two simple irreducible infinities is such that

$$\phi(z) - G\left(\frac{1}{z - \alpha_1} - \frac{1}{z - \alpha_2}\right)$$

is finite everywhere within the parallelogram, provided G be a properly chosen constant.

XIII. Two functions $\phi(z)$ and $\psi(z)$, which are doubly periodic in the same periods and have the same two simple irreducible infinities, are equivalent to one another.

For G and H can be so chosen that

$$\phi(z) - G\left(\frac{1}{z - \alpha_1} - \frac{1}{z - \alpha_2}\right)$$

and

$$\psi(z) - H\left(\frac{1}{z - \alpha_1} - \frac{1}{z - \alpha_2}\right)$$

are finite everywhere within the parallelogram. Hence $H\phi(z) - G\psi(z)$ is finite everywhere within the parallelogram and, being a doubly periodic function, is so everywhere. Hence by VI it is a constant; and therefore

$$\phi(z) = C\psi(z) + C'$$

where C and C' are constants.

Corollary. If the functions also have the same zeros, then

$$\phi(z) = C\psi(z).$$

XIV. In the same way it may be proved that—

(i) two functions ϕ and ψ, doubly periodic in the same periods and having the same single irreducible infinity of the second order, are equivalent to one another; and

(ii) two functions doubly periodic in the same periods having the same single irreducible infinity of order n and such that for each of them the quantities $G_2, G_3, \dots, G_{n-1}$ are zero (when there is therefore in their expansion only a single term which is ultimately infinite) are equivalent to one another.

Weierstrass's function σ.

4. In the functions which occur there are two periods which may be denoted by 2ω and $2\omega'$, where

$$\omega = \omega_1 + i\nu_1, \text{ and } \omega' = \omega_2 + i\nu_2,$$

and the quantities ω_1, ω_2, ν_1, ν_2 are real. We write for shortness

$$\Omega = 2m\omega + 2m'\omega'.$$

Then the definition of the σ function is

$$\sigma(u) = u\Pi'\left(1 + \frac{u}{\Omega}\right) e^{\frac{u}{\Omega} + \frac{1}{2}\frac{u}{\Omega^2}},$$

where Π' implies that the product is to be taken for the factors arising from all integral values of m and of m' from $+\infty$ to $-\infty$, except the double-zero combination. It is assumed that the numerical value of the negative infinite limit for m is the same as that of the positive infinite limit, these being $\pm p$; and likewise for m', these being $\pm q$. But no relation is assumed to exist between p and q.

Evidently $\sigma(u)$ is an uneven function; for associating in the product the two terms, which arise from taking the factor determined by $2m\omega + 2m'\omega'$ with that determined by $-2m\omega - 2m'\omega'$, we have

$$\sigma(u) = \quad u\Pi'\left(1 - \frac{u^2}{\Omega^2}\right) e^{\frac{u^2}{\Omega^2}}.$$

Hence $$\sigma(-u) = -u\Pi'\left(1 - \frac{u^2}{\Omega^2}\right) e^{\frac{u^2}{\Omega^2}}$$

$$= -\sigma(u).$$

5. From the original definition we have

$$\log \sigma(u) = \log u + \Sigma'\left\{\frac{u}{\Omega} + \frac{1}{2}\frac{u^2}{\Omega^2} + \log\left(1 + \frac{u}{\Omega}\right)\right\}$$

where Σ' implies summation for the same combinations of m and m' as occur in the products; from this we have

$$\frac{\sigma'(u)}{\sigma(u)} = \frac{1}{u} + \Sigma'\left\{\frac{1}{\Omega} + \frac{u}{\Omega^2} + \frac{1}{u + \Omega}\right\}.$$

6. The value of the left-hand side, when to u is assigned the value ω, is denoted by η; when to u is assigned the value ω', it is denoted by η'. Thus we have

$$\eta = \frac{1}{\omega} + \Sigma' \left\{ \frac{1}{\Omega} + \frac{\omega}{\Omega^2} + \frac{1}{\omega + \Omega} \right\}$$

$$= \Sigma' \frac{1}{\Omega} + \omega \Sigma' \frac{1}{\Omega^2} + \Sigma \frac{1}{\omega + \Omega}$$

where Σ implies summation for all integral combinations of m and m' within their limits *including* the double-zero combination. In the sum $\Sigma' \frac{1}{\Omega}$, for every term Ω^{-1} that enters there is a corresponding term $-\Omega^{-1}$ arising from the values of m and m' opposite in sign to those which determine Ω^{-1}. As the limits for m are $\pm p$, where p is infinite, and those for m' are $\pm q$, where q is infinite, it follow that $\Sigma' \Omega^{-1}$ is zero. Hence

$$\eta = \omega \; \Sigma' \frac{1}{\Omega^2} + \Sigma \frac{1}{\omega + \Omega}.$$

Similarly

$$\eta' = \omega' \; \Sigma' \frac{1}{\Omega^2} + \Sigma \frac{1}{\omega' + \Omega}.$$

Hence

$$\eta\omega' - \eta'\omega = \omega' \sum_{m=-p}^{m=p} \sum_{m'=-q}^{m'=q} \frac{1}{(2m+1)\,\omega + 2m'\omega'}$$

$$- \omega \sum_{m=-p}^{m=p} \sum_{m'=-q}^{m'=q} \frac{1}{2m\omega + (2m'+1)\,\omega'}.$$

Consider now the first of these sums, and denote it by S. We have

$$S = \sum_{m'=-q}^{m'=q} \frac{1}{(2p+1)\,\omega + 2m'\omega'}$$

$$+ \sum_{r=1}^{r=p} \sum_{m'=-q}^{m'=q} \left\{ \frac{1}{(2r-1)\,\omega + 2m'\omega'} - \frac{1}{(2r-1)\,\omega - 2m'\omega'} \right\}.$$

But $\quad (2r-1)\,\omega + 2k\omega' = (2r-1)\,\omega - 2\,(-k)\,\omega'$;

hence the second line vanishes when we select corresponding terms, and therefore

$$S = \sum_{m'=-q}^{m'=q} \frac{1}{(2p+1)\,\omega + 2m'\omega'}.$$

In this p is infinite, hence we may write

$$S = \int_{-q}^{q} \frac{dm'}{(2p+1)\,\omega + 2m'\omega'}.$$

Now, if $L(u)$ denote the principal logarithm of u, when there is a principal value, that is, when the real part of u is positive, and writing $L_{\pm\pi i}(u)$ to denote $L(u)$ or $L(-u) \pm \pi i$ according as the real part of u is positive or negative we have, by a theorem of Cayley's,*

$$S = \frac{1}{2\omega'} L_{\pm\pi i} \frac{(2p+1)\,\omega + 2q\omega'}{(2p+1)\,\omega - 2q\omega'}$$

$$= \frac{1}{2\omega'} L_{\pm\pi i} \frac{p\omega + q\omega'}{p\omega - q\omega'}$$

(since p and q are infinite), and the positive or negative sign is taken according as $\omega_1 v_2 - \omega_2 v_1$ is positive or negative.

Denoting the second sum by S', we find in the same way that

$$S' = \frac{1}{2\omega} L_{\mp\pi i} \frac{q\omega' + p\omega}{q\omega' - p\omega},$$

the doubtful sign being determined as before. Thus

$$2(\eta\omega' - \eta'\omega) = L_{\pm\pi i} \frac{p\omega + q\omega'}{p\omega - q\omega'} - L_{\mp\pi i} \frac{q\omega' + p\omega}{q\omega' - p\omega}.$$

Let
$$\frac{p\omega + q\omega'}{p\omega - q\omega'} = X + Yi.$$

Then in the case when X is positive we have

$$L_{\pm\pi i} \frac{p\omega + q\omega'}{p\omega - q\omega'} = L(X + Yi),$$

and
$$L_{\mp\pi i} \frac{q\omega' + p\omega}{q\omega' - p\omega} = L(X + Yi) \mp \pi i;$$

and therefore in this case

$$2(\eta\omega' - \eta'\omega) = \pm\pi i.$$

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* *Liouville*, 1ème Sér., t. x., p. 419.

In the case when X is negative we have

$$L_{\pm\pi i}\frac{p\omega+q\omega'}{p\omega-q\omega'}=L(-X-Yi)\pm\pi i,$$

$$L_{\mp\pi i}\frac{q\omega'+p\omega}{q\omega'-p\omega}=L(-X-Yi);$$

and therefore as before

$$2(\eta\omega'-\eta'\omega)=\pm\pi i.$$

Hence, in general, we have this as our result, the upper or lower sign being taken according as $\omega_1 v_2-\omega_2 v_1$ is positive or negative, or what is the same thing, according as the real part of $\omega'/\omega i$ is positive or negative.

Weierstrass's function $P(u)$.

7. This is defined to be the value of $-\frac{d^2}{du^2}\{\log\sigma(u)\}$. It is an even function, since $\sigma(u)$ is an odd function.

From the former value of $\sigma'(u)/\sigma(u)$ we have

$$-P(u)=-\frac{1}{u^2}+\Sigma'\left\{\frac{1}{\Omega^2}-\frac{1}{(u+\Omega)^2}\right\},$$

so that
$$P(u)=\Sigma\frac{1}{(u+\Omega)^2}-\Sigma'\frac{1}{\Omega^2},$$

where Σ and Σ' have the same signification as before.

Periodicity of $P(u)$.

8. We have

$$P(u+2\omega)-P(u)$$

$$=\sum_{m=-p}^{m=p}\sum_{m'=-q}^{m'=q}\frac{1}{(u+2\omega+2m\omega+2m'\omega')^2}-\sum_{m=-p}^{m=p}\sum_{m'=-q}^{m'=q}\frac{1}{(u+2m\omega+2m'\omega')^2}$$

$$=\sum_{m'=-q}^{m'=q}\frac{1}{\{u+(2p+2)\omega+2m'\omega'\}^2}-\sum_{m'=-q}^{m'=q}\frac{1}{(u-2p\omega+2m'\omega')^2}$$

$$=\int_{-q}^{q}\frac{dm'}{\{u+(2p+2)\omega+2m'\omega'\}^2}-\int_{-q}^{q}\frac{dm'}{(u-2p\omega+2m'\omega')^2},$$

since p is infinite. But, since p and q are infinite, each of these integrals is of the form

$$\frac{1}{2\omega'}\left[\frac{1}{\infty}-\frac{1}{\infty}\right],$$

and is therefore zero. Hence

$$\begin{aligned} P(u+2\omega) &= P(u) \\ &= P(u+2\omega') \end{aligned}$$

in the same way; and therefore $P(u)$ is doubly periodic in the two periods 2ω and $2\omega'$.

Obviously $P'(u+2\omega) = P'(u) = P'(u+2\omega')$;

and therefore, as $P'(u)$ is an odd function, we have

$$P'(\omega) = 0 = P'(\omega') = P'(\omega+\omega').$$

It is convenient to write

$$\omega + \omega' = \omega'',$$

so that $$P'(\omega) = 0 = P'(\omega') = P'(\omega'').$$

Quasi-Periodicity of $\sigma(u)$.

9. We have

$$\begin{aligned} \frac{d^2}{du^2}\{\log\sigma(u+2\omega)\} &= -P(u+2\omega) \\ &= -P(u) \\ &= \frac{d^2}{du^2}\{\log\sigma(u)\}; \end{aligned}$$

hence $$\frac{\sigma'(u+2\omega)}{\sigma(u+2\omega)} = \frac{\sigma'(u)}{\sigma(u)} + A,$$

where A is independent of u. To determine A we write $u = -\omega$; whence, by noticing that $\sigma'(u)/\sigma(u)$ is an odd function, we see that

$$A = 2\frac{\sigma'(\omega)}{\sigma(\omega)} = 2\eta;$$

and therefore

$$\frac{\sigma'(u+2\omega)}{\sigma(u+2\omega)} = \frac{\sigma'(u)}{\sigma(u)} + 2\eta.$$

Similarly we find

$$\frac{\sigma'(u+2\omega')}{\sigma(u+2\omega')} = \frac{\sigma'(u)}{\sigma(u)} + 2\eta'.$$

Combining these results we have

$$\frac{\sigma'(u+2m\omega+2m'\omega')}{\sigma(u+2m\omega+2m'\omega')} = \frac{\sigma'(u)}{\sigma(u)} + 2m\eta + 2m'\eta'.$$

By writing $u = \omega' - \omega$ in the first, we have

$$\frac{\sigma'(\omega'')}{\sigma(\omega'')} = \frac{\sigma'(\omega'-\omega)}{\sigma(\omega'-\omega)} + 2\eta;$$

and, by writing $u = \omega - \omega'$ in the second, we have

$$\frac{\sigma'(\omega'')}{\sigma(\omega'')} = \frac{\sigma'(\omega-\omega')}{\sigma(\omega-\omega')} + 2\eta'.$$

From these we have, by addition,

$$\frac{\sigma'(\omega'')}{\sigma(\omega'')} = \eta + \eta';$$

and, as it is convenient to write

$$\eta'' = \eta + \eta',$$

this gives $$\eta'' = \frac{\sigma'(\omega'')}{\sigma(\omega'')}.$$

10. Integrating the equation

$$\frac{\sigma'(u+2\omega)}{\sigma(u+2\omega)} = \frac{\sigma'(u)}{\sigma(u)} + 2\eta,$$

we have $$\sigma(u+2\omega) = Be^{2\eta u}\sigma(u),$$

where B is a constant of integration which may be determined by writing $u = -\omega$, and then

$$\sigma(\omega) = Be^{-2\eta\omega}\sigma(-\omega),$$

so that $$B = -e^{2\eta\omega};$$

and therefore $$\sigma(u+2\omega) = -e^{2\eta(u+\omega)}\sigma(u).$$

From this it follows that

$$\sigma(u+4\omega) = -e^{2\eta(u+2\omega+\omega)}\sigma(u+2\omega)$$
$$= e^{4\eta(u+2\omega)}\sigma(u);$$

and generally

$$\sigma(u+2m\omega) = (-1)^m e^{2m\eta(u+m\omega)}\sigma(u).$$

Similarly

$$\sigma(u+2m'\omega') = (-1)^{m'} e^{2m'\eta'(u+m'\omega')}\sigma(u).$$

Hence

$$\sigma(u+2m\omega+2m'\omega')$$
$$= (-1)^{m'} e^{2m'\eta'(u+2m\omega+m'\omega')}\sigma(u+2m\omega)$$
$$= (-1)^{m+m'} e^{2m'\eta'(u+2m\omega+m'\omega')+2m\eta(u+m\omega)}\sigma(u)$$
$$= (-1)^{m+m'} e^{(2m\eta+2m'\eta')(u+m\omega+m'\omega')+2mm'(\eta'\omega-\eta\omega')}\sigma(u).$$

But $\qquad 2(\eta'\omega-\eta\omega') = \mp\pi i,$

and therefore $\qquad e^{2(\eta'\omega-\eta\omega')} = -1;$

hence

$$\sigma(u+2m\omega+2m'\omega') = (-1)^{mm'+m+m'} e^{(2m\eta+2m'\eta')(u+m\omega+m'\omega')}\sigma(u),$$

which is the law of quasi-periodicity of the σ function.

11. Using our partial results in the reverse order, we find

$$\sigma(u+2m\omega+2m'\omega')$$
$$= (-1)^{m+m'} e^{(2m\eta+2m'\eta')(u+m\omega+m'\omega')+2mm'(\eta\omega'-\eta'\omega)}\sigma(u);$$

and therefore from a division of corresponding members of the two equations

$$e^{4mm'(\eta'\omega-\eta\omega')} = 1,$$

from which it is to be inferred that $4(\eta'\omega-\eta\omega')$ is some positive or negative integral multiple of $2\pi i$. This is merely an indirect verification of the result already obtained for $\eta\omega'-\eta'\omega$, and is not sufficient to determine its value.

The differential equation satisfied by $\sigma(u)$.

12. From § 5 we have

$$\log \sigma(u) = \log u + \Sigma' \left\{ \frac{u}{\Omega} + \tfrac{1}{2}\frac{u^2}{\Omega^2} + \log\left(1 + \frac{u}{\Omega}\right) \right\}$$

$$= \log u + \Sigma' \left\{ \frac{u}{\Omega} + \tfrac{1}{2}\frac{u^2}{\Omega^2} + \frac{u}{\Omega} - \tfrac{1}{2}\frac{u^2}{\Omega^2} + \tfrac{1}{3}\frac{u^3}{\Omega^3} - \tfrac{1}{4}\frac{u^4}{\Omega^4} + \ldots \right\}.$$

Now $\Sigma'\Omega^{-2n+1} = 0$, so that

$$\log \sigma(u) = \log u - \tfrac{1}{4}u^4\Sigma' \frac{1}{\Omega^4} - \tfrac{1}{6}u^6\Sigma' \frac{1}{\Omega^6} - \ldots.$$

It is convenient to write

$$g_2 = 2^2.3.5\,\Sigma' \frac{1}{\Omega^4},$$

$$g_3 = 2^2.5.7\,\Sigma' \frac{1}{\Omega^6};$$

so that the constants g_2 and g_3 are functions of ω and ω'; they are fundamental in the theory. Conversely ω and ω' are functions of g_2 and g_3; and therefore $\Sigma'\Omega^{-2n}$ is a function of g_2 and g_3; and this function is found afterwards to be a rational algebraical integral function.

Then $\sigma(u)$ satisfies the partial differential equation

$$\frac{\partial^2\sigma}{\partial u^2} = 12g_3\frac{\partial\sigma}{\partial g_2} + \tfrac{2}{3}g_2^{\,2}\frac{\partial\sigma}{\partial g_3} - \tfrac{1}{12}g_2 u^2\sigma.$$

A proof of this occurs in a memoir by Simon, *Crelle*, t. LXXXI., p. 311, and so need not be here reproduced. By means of the equation the expansion of $\sigma(u)$ in powers of u is obtained by Weierstrass, and expressions for the sums of the inverse powers $\Sigma\Omega^{-2n}$ can be derived.*

The differential equation satisfied by $P(u)$.

13. From § 7 we have

$$P(u) = \Sigma' \frac{1}{(u+\Omega)^2} - \Sigma' \frac{1}{\Omega^2} + \frac{1}{u^2}$$

$$= \frac{1}{u^2} + \Sigma' \frac{1}{\Omega^2}\left(-2\frac{u}{\Omega} + 3\frac{u^2}{\Omega^2} - 4\frac{u^2}{\Omega^3} + \ldots\right)$$

$$= \frac{1}{u^2} + 3u^2\Sigma' \frac{1}{\Omega^4} + 5u^4\Sigma' \frac{1}{\Omega^6} + \ldots.$$

* See some expressions similar to these in the *Quart. Journ. Math.*, vol. XXI., pp. 261—230.

so that $P(u)$ has a single irreducible infinity of the second order for $u=0$, and is therefore of the type of function considered in X., § 3.

Again $P^2(u)$ has a single infinity of the fourth order; and the infinite value arises from only a single term. Hence $P^2(u)$ is a function of the type considered in XIV., § 3.

Since $P(u)$ is doubly periodic in 2ω and $2\omega'$, so also are all its differential coefficients. Also we have

$$P''(u) = \frac{6}{u^4} + 6\,\Sigma'\frac{1}{\Omega^4} + 3.4.5\,u^2\,\Sigma'\frac{1}{\Omega^6} + \ldots.$$

Hence $P^2(u)$ and $P''(u)$ are two doubly-periodic functions with the same periods; they have the same single irreducible infinity of the same order, and in each the terms which give the infinite value are single. Hence the functions are of the type of XIV., § 3, and therefore by the second theorem in XIV., § 3, we have

$$P''(u) = 6P^2(u) + C',$$

where C and C' are constants. These constants are easily determined from a comparison of different terms, with the result

$$P''(u) = 6P^2(u) - \tfrac{1}{2}g_2.$$

Hence $$P'^2(u) = 4P^3(u) - g_2P(u) - g_3,$$

the constant of integration being determined from a comparison of terms. This is the differential equation satisfied by $P(u)$.

Now, $P'(u)$ vanishes when $u=\omega$, or ω', or ω''; hence

$$\left\{\frac{dP(u)}{du}\right\}^2 = 4\,\{P(u)-P(\omega)\}\,\{P(u)-P(\omega')\}\,\{P(u)-P(\omega'')\}.$$

Let x, e_1, e_2, e_3 denote $P(u)$, $P(\omega)$, $P(\omega'')$, $P(\omega')$ respectively; then, noticing that $P(u)$ is infinite when u is zero, we have

$$2u = \int_x^\infty \frac{dx}{\{(x-e_1)(x-e_2)(x-e_3)\}^{\frac{1}{2}}},$$

or $$u = \int_x^\infty \frac{dx}{(4x^3 - g_2x - g_3)^{\frac{1}{2}}},$$

and the expression of x as an elliptic function of u is easily obtained.

[14. Though not connected with the theory it may not be out of place to state the reduction of the general elliptic integral to this as a canonical form. Let

$$U = (a, b, c, d, e \between x, y)^4;$$

H be the Hessian of U, so that

$$H = (ac - b^2, \ldots \between x, y)^4;$$

Φ the cubicovariant, so that

$$\Phi = \tfrac{1}{8}\left(\frac{\partial U}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial x}\frac{\partial U}{\partial y}\right)$$

$$= (a^2d - 3abc + 2b^3, \ldots \between x, y)^6;$$

and let I and J be the quadrinvariant and cubinvariant respectively. Then writing

$$z = -\frac{H}{U},$$

and making y unity, we have

$$\frac{dx}{\{(a, b, c, d, e \between x, 1)^4\}^{\frac{1}{2}}} = \frac{1}{2}\,\frac{dz}{(4z^3 - Iz - J)^{\frac{1}{2}}}.]^*$$

A general doubly periodic-function.

15. Consider a function

$$\phi(u) = A\,\frac{\sigma(u - a_1)\,\sigma(u - a_2)\ldots\sigma(u - a_n)}{\sigma(u - b_1)\,\sigma(u - b_2)\ldots\sigma(u - b_n)},$$

where the quantities A, a, b are independent of u.
Then

$$\phi(u + 2\omega) = A\,\frac{\sigma(u + 2\omega - a_1)\ldots\sigma(u + 2\omega - a_n)}{\sigma(u + 2\omega - b_1)\ldots\sigma(u + 2\omega - b_n)}$$

$$= e^{(2m\eta + 2m'\eta')(\Sigma b - \Sigma a)}\phi(u)$$

$$= \phi(u + 2\omega');$$

so that $\phi(u)$ is a doubly periodic function, with periods 2ω and $2\omega'$, provided

$$\Sigma b = \Sigma a.$$

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* See Faà de Bruno, *Amer. Journal of Math.*, t. 5, pp. 1–24.

Addition Theorem for the σ-function.

16. For the σ-function there is no addition theorem proper, that is to say, $\sigma(u+v)$ is not expressible in terms of functions of u and v alone; but the value of $\sigma(u+v)\,\sigma(u-v)$ is so expressible. The analogy with the theta functions is obvious.

Consider a function

$$A\,\frac{\sigma(u-v)\,\sigma(u+v)}{\sigma^2(u)}.$$

By the last paragraph this function is, *quà* function of u, doubly periodic, and has for its periods 2ω and $2\omega'$. Also we have, by § 12,

$$\sigma(u) = ue^{-\frac{1}{240}g_2u^4 - \frac{1}{840}g_3u^6 - \dots};$$

and therefore
$$\frac{1}{\sigma(u)} = \frac{1}{u}e^{\frac{1}{240}g_2u^4 + \frac{1}{840}g_3u^6 + \dots}$$
$$= \frac{1}{u} + \tfrac{1}{240}g_2u^3 + \dots .$$

The function considered has thus a single irreducible infinity of the second order, given by $u=0$. Its periodicity and infinity are the same as those of $P(u)$, and therefore by § 3, XIV (i)

$$A\,\frac{\sigma(u-v)\,\sigma(u+v)}{\sigma^2(u)} = EP(u) - F,$$

where A, E, F are independent of u. Now the left-hand side vanishes if $u=v$; hence

$$F = EP(v).$$

When we write $A = \dfrac{1}{\sigma^2(v)}$, which satisfies the only condition assigned, the equation takes the form

$$\frac{\sigma(u-v)\,\sigma(u+v)}{\sigma^2(u)\,\sigma^2(u)} = E\{P(u) - P(v)\}.$$

The skew symmetry of each side shews that E, being independent of u, is also independent of v, and therefore is a constant. To determine it we consider terms on each side; retaining those of lowest indices, we have

$$\frac{u^2 - v^2}{u^2v^2} = E\left(\frac{1}{u^2} - \frac{1}{v^2}\right),$$

so that $E = -1$. Hence

$$\frac{\sigma(u-v)\,\sigma(u+v)}{\sigma^2(u)\,\sigma^2(v)} = P(v) - P(u),$$

which is the fundamental equation in the development of the theory of the functions.

In a similar manner it may be proved that

$$P'(u) = -2\frac{\sigma(u-\omega)\,\sigma(u-\omega')\,\sigma(u+\omega'')}{\sigma(\omega)\,\sigma(\omega')\,\sigma(\omega'')\,\sigma^3(u)}$$

$$= 2\frac{\sigma(u+\omega)\,\sigma(u+\omega')\,\sigma(u-\omega'')}{\sigma(\omega)\,\sigma(\omega')\,\sigma(\omega'')\,\sigma^3(u)},$$

from which, by the help of the foregoing equation, that of § 13 can be constructed.

The functions σ_1, σ_2, σ_3.

17. The functions σ_1, σ_2, σ_3 are defined by the relations

$$\sigma_1(u) = e^{-u\eta}\frac{\sigma(u+\omega)}{\sigma(\omega)},$$

$$\sigma_2(u) = e^{-u\eta''}\frac{\sigma(u+\omega'')}{\sigma(\omega'')},$$

$$\sigma_3(u) = e^{-u\eta'}\frac{\sigma(u+\omega')}{\sigma(\omega')}.$$

We proceed to obtain for these functions the product expressions similar to that which is given as the definition of $\sigma(u)$. From the definition of $\sigma_1(u)$ we have

$$\frac{\sigma_1'(u)}{\sigma_1(u)} = -\eta + \frac{\sigma'(u+\omega)}{\sigma(u+\omega)}$$

$$= -\frac{\sigma'(\omega)}{\sigma(\omega)} + \frac{\sigma'(u+\omega)}{\sigma(u+\omega)}$$

$$= -\frac{1}{\omega} - \Sigma'\left\{\frac{1}{\Omega} + \frac{\omega}{\Omega^2} + \frac{1}{\omega-\Omega}\right\}$$

$$+ \frac{1}{u+\omega} - \Sigma'\left\{\frac{1}{\Omega} + \frac{u+\omega}{\Omega^2} + \frac{1}{u+\omega-\Omega}\right\}$$

$$= u\Sigma'\frac{1}{\Omega^2} + \Sigma\left\{\frac{1}{u+\omega-\Omega} - \frac{1}{\omega-\Omega}\right\}.$$

Also $$e_1 = P(\omega)$$

$$= \Sigma\frac{1}{(\omega-\Omega)^2} - \Sigma'\frac{1}{\Omega^2};$$

hence

$$\frac{\sigma_1'(u)}{\sigma_1(u)} + e_1 u = \Sigma \left\{ \frac{u}{(\omega - \Omega)^2} + \frac{1}{u + \omega - \Omega} - \frac{1}{\omega - \Omega} \right\}$$

$$= \Sigma \left\{ \frac{u}{\Omega_1^{\,2}} + \frac{1}{u - \Omega_1} + \frac{1}{\Omega_1} \right\}$$

$$= \frac{d}{du} \log \left\{ \Pi \left(1 - \frac{u}{\Omega_1}\right) e^{\frac{u}{\Omega_1} + \frac{1}{2}\frac{u^2}{\Omega_1^{\,2}}} \right\},$$

where $$\Omega_1 = \Omega - \omega$$

$$= (2m - 1)\,\omega + 2m'\omega'.$$

Hence

$$\sigma_1(u) = Ae^{-\frac{1}{2}e_1u^2}\, \Pi \left(1 - \frac{u}{\Omega_1}\right) e^{\frac{u}{\Omega_1} + \frac{1}{2}\frac{u^2}{\Omega_1^{\,2}}}.$$

From the definition of $\sigma_1(u)$, it at once follows that $\sigma_1(0) = 1$, and therefore $A = 1$; thus

$$\sigma_1(u) = e^{-\frac{1}{2}e_1u^2}\, \Pi \left(1 - \frac{u}{\Omega_1}\right) e^{\frac{u}{\Omega_2} + \frac{1}{2}\frac{u^2}{\Omega_1^{\,2}}}.$$

It is not difficult to see that, from the expansion of

$$\Pi \left(1 - \frac{u}{\Omega_1}\right) e^{\frac{u}{\Omega_1} + \frac{1}{2}\frac{u^2}{\Omega_1^{\,2}}},$$

in ascending powers of u, we may write

$$\Omega_1 = (2m + 1)\,\omega + 2m'\omega'.$$

Hence, taking this and defining

$$\Omega_2 = (2m + 1)\,\omega + (2m' + 1)\,\omega',$$

$$\Omega_3 = \qquad 2m\omega + (2m' + 1)\,\omega',$$

we have

$$\sigma_\lambda(u) = e^{-\frac{1}{2}e_\lambda u^2} \prod_{m=-p}^{m=p} \prod_{m'=-q}^{m'=q} \left(1 - \frac{u}{\Omega_\lambda}\right) e^{\frac{u}{\Omega_\lambda} + \frac{1}{2}\frac{u^2}{\Omega_\lambda^{\,2}}},$$

for $\lambda = 1, 2, 3$.

18. These equations form the basis of the theory; for the remainder, reference should be made to Weierstrass's lectures, where the development is given in somewhat fuller detail than in the introduction, an attempt to supply which is here made.

NUMERICAL SOLUTION OF A BIQUADRATIC EQUATION BY DESCARTES' PROCESS.

By *H. W. Lloyd Tanner, M.A.*

The following method of effecting the numerical solution of a biquadratic is convenient while more general than the method of divisors. The given biquadratic is brought to the form

$$x^4 + bx^3 + cx^2 + dx + e = 0,$$

the coefficients being integers. The left-hand member is to be broken up into quadratic factors,

$$(x^2 + px + q)(x^2 + p'x + q').$$

The proposed method consists in a process of calculating p, p', q, q' when they are commensurable; or, what is the same thing in the present case, when they are integers.

Since $$qq' = e,$$

we take for q, q' any pair of co-factors of e. If this pair proves unsuitable, another pair is selected; and so on until the required resolution is effected, or by exhausting the set we verify that there are no quadratic factors with commensurable coefficients.

For the determination of p corresponding to a trial pair q, q' we use the equation

$$p(q' - q) = d - bq.$$

showing that $d - bq$ must be divisible by $q' - q$.

For p' we have then

$$p' = b - p.$$

If the values thus found are suitable they must satisfy the relation

$$pp' = c - (q + q').$$

A convenient form of setting down the work is indicated in the example which follows.

Example. $x^4 + 6x^3 - 10x^2 - 60x + 72 = 0.$

1	$+6$	-10	-60	$+72$	
			-6.11	1.72	71
			-6.9	$-$	
			-6.12	2.36	34
			-6.8	$-$	
			-6.13	3.24	21
	$2+4$	17	-6.7	$-$	-21
	$-6+12$	-32	-6.14	4.18	14
			-6.6	$-$	
	$-16+22$	-28	-6.16	6.12	6
	$4+2$	8	-6.4	$-$	-6

therefore $(x^2 + 4x - 6)(x^2 + 2x - 12) = 0.$

The column under 72 (e) contains the pairs of co-factors of 72, as far as necessary. The arrangement adopted is defined by three conditions. (1) The smaller factor is taken for q. (2) The two pairs (q, q') and $(-q, -q')$ are tested in consecutive lines. In the first or "leading" line we take q' to be positive so that q is of the same sign as e. This gives $q' - q$ (written in the column to the right) a positive value in the leading lines. It will generally be unnecessary to fill these columns except for the leading lines. (3) The values of q are taken in ascending arithmetical order.

The column under -60 (d) contains the values of

$$-60 - 6q \; (= d - bq)$$

for the values of q in the several lines. Practically it is more convenient to consider these numbers as the values of $d \mp bq$ where q has its leading values; and it saves labour to express as a separate factor the G.C.M. of b and d. It is to be noted that if the biquadratic has no second term $(b = 0)$ this column is superfluous. But, as will be seen from the example worked below, it does not always pay to remove the second term by a preliminary reduction.

If the d element is divisible by the corresponding value of $q - q'$ the quotient is placed as a first element in the column under b. The sum of the two elements in any line of the b column is b.

If the b column is filled we write under c the result of subtracting the sum $q + q'$ from c. This is equal to the product of the b elements when the trial is successful. The first number under b and the first number under e are then the

values of p, q; the second numbers are p', q', in a product,

$$(x^2+px+q)(x^2+p'x+q'),$$

to which the given expression is equal.

The above example is not favourable to the method, since 72 has so many divisors and the required pair come late in order. In all cases I have tried, the work is much less than solving by means of a commensurable root of the reducing cubic.

Example 2. $x^4-4x^3-477x^2+3248x-480=0$.

1	-4	-477	$+3248$	-480	
			4.811	-1.480	481
			4.813		
			4.810	-2.240	242
			4.814		
			4.809	-3.160	163
	$-20+16$	-320	4.815	$3.-160$	-163

$$(x^2-20x+3)(x^2+16x-160).$$

Hence the roots are $10\pm\sqrt{(103)}$, $-8\pm4\sqrt{(14)}$.

The same example is now worked with a preliminary removal of the second term.

1	-4	-477	$+3248$	-480	
	-3	-480	$+2768$	$+2288$	
	-2	-482	$+2286$	$(=16.11.13)$	
	-1	-483	$(=2.9.127)$		
	0				
				1.2288	2287
				2.1144	1142
				4.572	568
				8.286	278
				11.208	197
				13.176	163
	$18-18$	-642		16.143	127
	$-18+18$	-324		$-$	-127

Hence a resolution is

$$\{(x-1)^2-18(x-1)-16\}\{(x-1)^2+18(x-1)-143\}$$
$$=(x^2-20x+3)(x^2+16x-160)$$

as before.

It is not necessary to write the second lines when the division test fails, since it must fail for both or neither of the lines (q, q'), $(-q, -q')$ if $b=0$. The expression of e in factors is useful to avoid the risk of omitting any q; and, when $b=0$, the factorization of d considerably facilitates the application of the division test.

The method becomes illusory when

$$q = q',\ d = bq.$$

But these imply

$$q = \pm\sqrt{(e)} = d/b,$$

the last member of which fixes the sign of q. p, p' are then the two roots of

$$p^2 - bp + c - 2q = 0,$$

Whenever e is a square, it is well to try this case first; for this only means observing whether or not b/d is a square root of e.

NOTE ON THE CLASSIFICATION OF SOME ALGEBRAICAL SERIES.

By Prof. *H. W. Lloyd Tanner, M.A.*

I HAVE found the following classification useful, and as it does not appear in the text-books, it may be worth noting in the *Messenger*.

I. Quasi-Arithmetic Series. The n^{th} term, a_n, is a rational integral function of n. The sum of n terms may be expressed by means of Δa_1, $\Delta^2 a_1$, &c.; or a_n may be divided into factorials which can be separately summed.

II. Quasi-Harmonic Series. a_n is a rational fraction of n; the denominator of which is, or can be made, a factorial expression. Then a_n can be expressed as a sum of inverse factorials, with or without direct factorials.

III. Quasi-Geometric Series. The n^{th} term involves n as an exponent.

(i) The n^{th} term $= a_n x^n$ where a_n is a rational integral function of n. To sum, multiply series repeatedly by $1 - x$.

(ii) Recurring Series. Multiply by scale of relation, &c.

The names, methods, and results correspond to those of the simplest series in each class.

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ON AN EXTENSION OF ORDINARY ALGEBRA.

By *A. B. Kempe, M.A., F.R.S.*

1. It will be convenient for brevity to term this extension the "new" algebra, the ordinary quantitative algebra being designated the "old" algebra.

2. In the new algebra we have "quantities," and operations performed on them, which may be termed "addition," "multiplication" &c., as in the old algebra.

3. The new algebraic product ab and sum $a+b$, of two quantities a and b are, when expressed in terms of the operations of the old algebra,

$$\frac{ab(i+z-u)-(a+b)iz+uiz}{ab-(a+b)u+u(i+z)-iz},$$

and

$$\frac{ab(2i-z)-(a+b)i^2+i^2z}{ab-(a+b)z+2iz-i^2},$$

respectively, where z, i, and u are three arbitrary quantities answering to the 0, ∞, and 1 of the old algebra; so that in the new algebra

$$zb=z, \qquad z+b=b,$$
$$ib=i, \qquad i+b=i,$$
$$ub=b.$$

4. Observe, that if we put $z=0$, $i=\infty$, $u=1$, the new product and sum ab and $a+b$ become identical with the old product and sum ab and $a+b$ respectively. The old algebra is in fact a special case of the new algebra, viz. one of a series of homographic transformations, of which the new algebra is the general case.

5. The associative, commutative, and distributive laws apply to the new addition and multiplication precisely as they do to the operations of the same names in the old algebra.

6. In fact the new algebra is not distinguishable from the old, until we come to deal with numbers, when the difference becomes apparent.

Thus, take $z=0$, $u=1$, $i=6$.

Then the new ab in terms of the old algebra is

$$\frac{5ab}{ab-(a+b)+6}, \qquad e.g.\ 2\times 2=\frac{10}{3}=3\tfrac{1}{3}.$$

7. And the new $a+b$ expressed in the old algebra is

$$12\left\{\frac{ab-3(a+b)}{ab-36}\right\}, \qquad e.g.\ 2+2=3.$$

8. Also, if in the old algebra $a+b=6$, then $ab=5$ in the new; or, if $ab=36$ in the old, $a+b=\infty$ in the new.

9. The origin of the new algebra can be readily shown. It is obtained by a general homographic transformation of all quantities. Thus if a', b', c', ... be the transformations of a, b, c, ..., so that $\infty'=i$, $0'=z$, $1'=u$, we have, in the old algebra,

$$a'=\frac{La+M}{Na+R},$$

where
$$i=\frac{L}{N}, \quad z=\frac{M}{R}, \quad u=\frac{L+M}{N+R},$$

i. e. we have
$$a'=\frac{i(z-u)a+z(u-i)}{(z-u)a+(u-i)} \quad \text{..............(1)},$$

whence
$$a=\left(\frac{u-i}{z-u}\right)\left(\frac{z-a'}{a'-i}\right) \quad \text{.............. (2)}.$$

10. The new product $a'b'$ in terms of the old algebra is $(ab)'$, which by (1) is

$$\frac{i(z-u)ab+z(u-i)}{(z-u)ab+(u-i)},$$

which by (2) equals

$$\frac{a'b'(z+i-u)-(a'+b')iz+ziu}{a'b'-(a'+b')u+uz+ui-iz} \quad \text{...... (3)}.$$

11. So the new sum $a'+b'$ in terms of the old algebra is $(a+b)'$, which by (1) is

$$\frac{i(z-u)(a+b)+z(u-i)}{(z-u)(a+b)+(u-i)},$$

which by (2) equals

$$\frac{a'b'(2i-z)-(a'+b')i^2+i^2z}{a'b'-(a'+b')z+2iz-i^2} \quad \text{........... (4)}.$$

12. Observe that if we put

$$(3) \equiv f(a', b', u, i, z),$$

we have

$$(4) \equiv f(a', b', z, i, i).$$

13. If we employ the numbers 1, 2, 3, &c., to represent the expressions u, $u+u$, $u+u+u$, &c., of the new algebra, then the new products and sums of those numbers will be the same as if they were the ordinary products and sums of ordinary numbers, *e.g.* $2 \times 2 = 4$, and $2+2=4$.

A NINE-LINE CONIC.

By *F. Morley*.

We know that (1) the feet of the four normals from a point O, whose coordinates are X, Y to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{..........................(i)},$$

lie on the rectangular hyperbola

$$a^2 \frac{X}{x} - b^2 \frac{Y}{y} = a^2 - b^2 \text{....................(ii)},$$

which (2) passes through O and the centre C of the conic, and (3) through the two points at ∞ in the directions of the axes.

The equation (ii) is not altered by the substitution of ma, mb for a, b; hence the hyperbole is the locus of feet of normals from O to all the conics

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = m^2;$$

and it cuts orthogonally that one of the series which passes through O, for the tangent to (ii) at X, Y is

$$a^2 \frac{x}{X} - b^2 \frac{y}{Y} = a^2 - b^2 \text{..................(iii)},$$

which is also normal to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{X^2}{a^2} + \frac{Y^2}{b^2} \text{ ....(iv)};$$

or (4) the tangent at O to (ii) is perpendicular to the polar of O with regard to (i).

Also the pole of (iii) with regard to (iv) lies on (ii). For it is given by

$$\frac{x}{a^2}\Big/\frac{a^2}{X} = -\frac{y}{b^2}\Big/\frac{b^2}{Y} = \frac{X^2}{a^2} + \frac{Y^2}{b^2}\Big/ a^2 - b^2,$$

or
$$\frac{X}{a^2}\Big/\frac{a^2}{x} = -\frac{Y}{b^2}\Big/\frac{b^2}{y} = \ldots;$$

and therfore
$$a^2\frac{X}{x} - b^2\frac{Y}{y} = a^2 - b^3.$$

This we may state: (5) the hyperbola meets its normal at O again on the line conjugate to its tangent at O with respect to (i).

Now reciprocate with respect to any point K. Then

(1) Given a conic U and a straight line L, we can find four points P, Q, R, S on L such that a tangent from each to U subtends a right angle at K.

(2) A conic V can be drawn touching these four tangents, the line L, and the polar of K with respect to U.

(3) To the points at ∞ on the axes correspond two perpendicular lines through K conjugate with respect to U. These are touched by V, and of course it follows that the director circle of V goes through K. The two lines are constructed at once if K be outside U by drawing the tangents KE, KF and bisecting the angles between them. For the bisecting lines are perpendicular, and they divide EF harmonically, so that the polar of each lies on the other.

(4) The point T where V touches L, and the pole of L with regard to U, subtend a right angle at K.

(5) The reciprocal of the normal at O is a point N on L such that TN subtends a right angle at K, or from (4) it is the intersection of AK and L; the reciprocal of the line through the centre conjugate to the tangent at O is a point on the polar of K with regard to U, conjugate to T, or it is the intersection of the polars of K and T, that is, the pole of KT with regard to U. Hence the other tangent to V from N passes through the pole of KT.

Thus we have nine well-marked lines which touch the conic V.

We make V a circle by taking for K one of the foci of the hyperbola.

Let K and O coincide, so that L is the line ∞; then the facts may be stated:

(2) The tangents at the feet of the four normals from O to a conic, the polar of O, and (3) the two perpendicular conjugate lines through O, are all touched by a parabola.

Let C be the centre of the conic. Then (4) the axis of the parabola is perpendicular to CO, or since perpendicular tangents are drawn from O, CO is the directrix.

(5) N becomes the point at ∞ on CO, and if M be the pole of the line through O perpendicular to CO, M is on the tangent at the vertex.

Hence the parabola is easily constructed. For a line parallel to CO and at the same distance from M will contain the focus; and since the polar of O is a tangent the perpendicular to it through M will also contain the focus.

ANALYTICAL GEOMETRICAL NOTE ON THE CONIC.

By Prof. *Cayley*.

Take (X, Y, Z) the coordinates of a point on the conic $yz+zx+xy=0$, so that $YZ+ZX+XY=0$; clearly (Y, Z, X) and (Z, X, Y) are the coordinates of two other points on the same conic; and I say that the three points are the vertices of a triangle circumscribed about the conic

$$x^2+y^2+z^2-2yz-2zx-2xy=0.$$

In fact the equation of one of the sides is

$$\begin{vmatrix} x, & y, & z \\ X, & Y, & Z \\ Y, & Z, & X \end{vmatrix} = 0,$$

say this is $AX+BY+CZ=0$, where

$$A, B, C = XY-Z^2,\ YZ-X^2,\ XZ-Y^2;$$

and the condition in order that this side may touch the conic $x^2+y^2+z^2-2yz-2zx-2xy=0$ is $BC+CA+AB=0$.

But we have

$$\begin{aligned} BC+CA+AB = {} & Y^2Z^2+Z^2X^2+X^2Y^2 \\ & -X(Y^3+Z^3)-Y(Z^3+X^3)-Z(X^3+Y^3) \\ & +X^2YZ+XY^2Z+XYZ^2 \\ = {} & (YZ+ZX+XY)(-X^2-Y^2-Z^2+YZ+ZX+XY)=0, \end{aligned}$$

and similarly for the other two sides. The point (X, Y, Z) is an arbitrary point on the conic $yz+zx+xy=0$; and we thus see that we have a singly infinite series of triangles each inscribed in this conic and circumscribed about the conic $x^2+y^2+z^2-2yz-2zx-2xy=0$.

END OF VOL. XV.